Reachability Predictive Control: A Novel Control Method for Systems with Unknown Dynamics

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Preliminary Exam Presentation

Motivation

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  - Onboard learning methods
  - Known physical laws
  - Observable behaviors like previous trajectories

Existing Control Methods for Uncertain Systems

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  - Develop control action for worst case unknown disturbances
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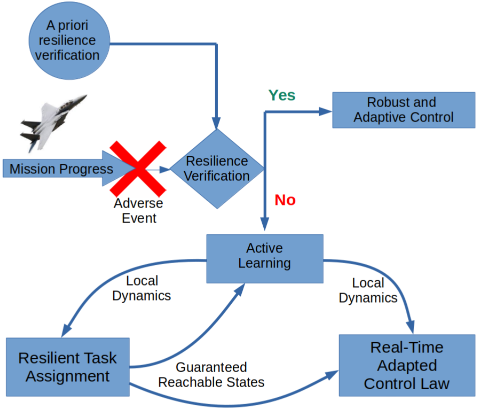
Control Pipeline

- Active learning
  - Learn local dynamics with an arbitrarily small error from test control inputs



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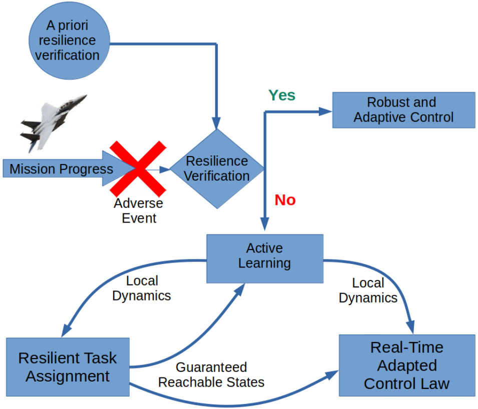

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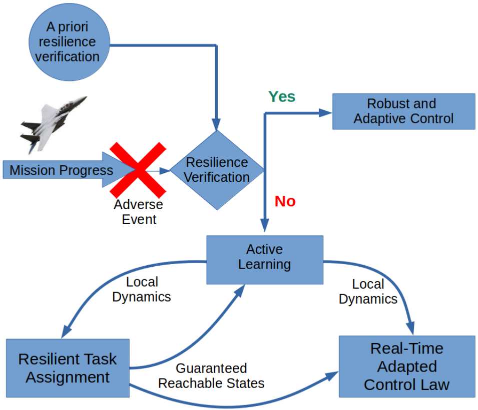

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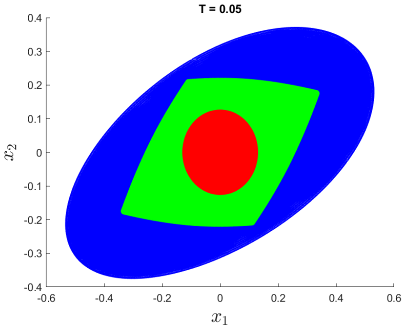
    - Use a proxy system model derived during resilient task assignment to synthesize a controller for small time intervals

Guaranteed Reachability

- It is impossible to determine the exact set of reachable states without full knowledge of the system dynamics

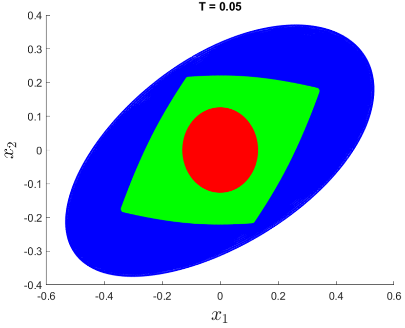


Guaranteed Reachability

- It is impossible to determine the exact set of reachable states without full knowledge of the system dynamics
- To determine what is provably possible, we want to underapproximate such a set

Guaranteed Reachable Set

A set of states that are provably achievable for a system within a given time frame



Framework for Under-Approximating the GRS

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Proposition

The reachable set of the proxy system is contained in the true reachable set of the unknown system for all time

Formalized Problem Statement

- We have an unknown nonlinear control-affine system of the form:
  - $\dot{x} = f(x) + G(x)\mathcal{U}, \quad x(0) = x_0$
  - $f(x) \in \mathbb{R}^n$ and $G(x) \in \mathbb{R}^{n \times m}$

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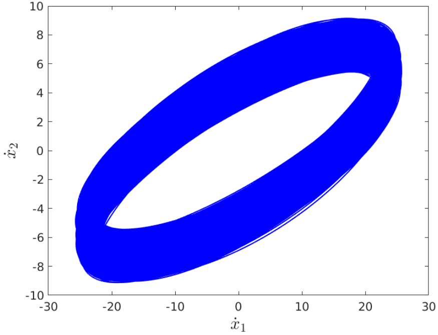
  - The input set $\mathcal{U} = \mathbb{B}^m(0; 1)$
  - Local dynamics $f(x_0)$ and $G(x_0)$
    - Learned within an arbitrarily small error from test control inputs
  - The maximum growth rate of dynamics given by Lipschitz bounds L_f and L_G
    - Determined from known physical laws
    - Uncertainty quantification

Problem Statement

Determine or underapproximate the guaranteed reachable set

Guaranteed Velocity Set

- We are left with infinitely many candidate systems consistent with our assumptions



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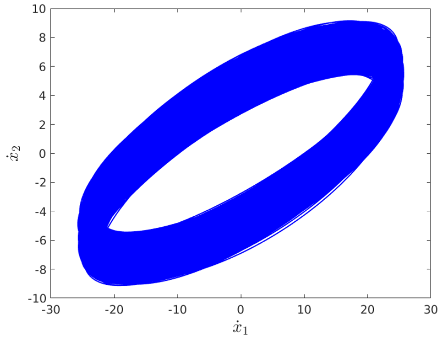

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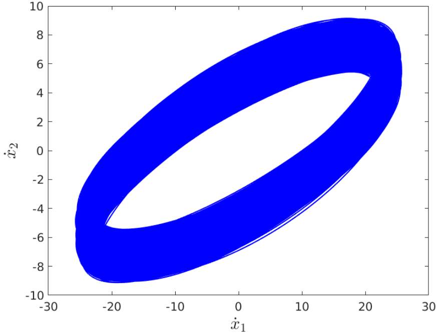
- Let us denote $\mathcal{D}_{con}$ as the set of all f, G consistent with our assumptions
- We introduce the following ODI:
  - $\dot{x} \in \mathcal{V}_x = f(x) + G(x)\mathcal{U}, \quad x(0) = x_0$

Guaranteed Velocity Set

The intersection of the set of all velocities whose dynamics are consistent with our assumptions:

$$\mathcal{V}_x^{\mathcal{G}} = \cap_{(f,G)\in\mathcal{D}_{con}} f(x) + G(x)\mathcal{U} \subset \mathcal{V}_x$$



Derived Ball Underapproximation

Ball Underapproximation

Let $\mathcal{U}$, L_f, and L_G be defined as above. Let $x \in \mathbb{R}^n$ satisfy $(L_f + L_G)\|x\| < \|G^\dagger(x_0)\|^{-1}$. Define

$$\bar{\mathcal{V}}_x^{\mathcal{G}} = \mathbb{B}^n(f(x_0); \|G^\dagger(x_0)\|^{-1} - (L_f + L_G)\|x\|) \cap \mathrm{Im}(G(x_0)).$$

Then, $\bar{\mathcal{V}}_x^{\mathcal{G}} \subseteq \mathcal{V}_x^{\mathcal{G}}$.

Derived Convex Underapproximation

Advanced Convex Underapproximation

Let $\mathcal{U}$, L_f, and L_G be defined as above. Let $\mu = 1$ if $\mathrm{rank}(G(x_0)) = m = n$, $\mu = \sqrt{2}$ if $\mathrm{rank}(G(x_0)) = \min(m, n)$ and $m \neq n$, $\mu = \frac{1+\sqrt{5}}{2}$ if $\mathrm{rank}(G(x_0)) < \min(m, n)$, and let x satisfy $(L_f + L_G)\|x\| \leq \|G^\dagger(x_0)\|^{-1}$. If

$$\bar{\bar{\mathcal{V}}}_x^{\mathcal{G}} = \{f(x_0) + kd \mid \|d\| = 1, d \in \mathrm{Im}(R), 0 \leq k \leq K(d)\}$$

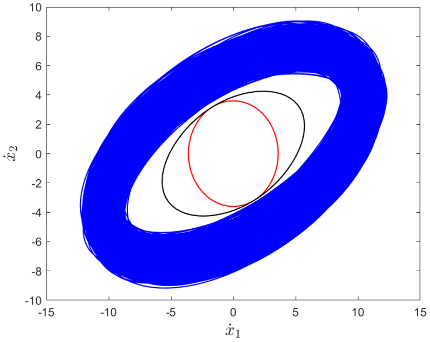
$$\text{s.t.} \quad K(d) = \frac{\|G^\dagger(x_0)\|^{-1} - (L_f + L_G)\|x\|}{\|G^\dagger(x_0)d\|(\|G(x_0)^\dagger\|^{-1} - L_G\|x\|) + \mu\|G(x_0)^\dagger\|L_G\|x\|},$$

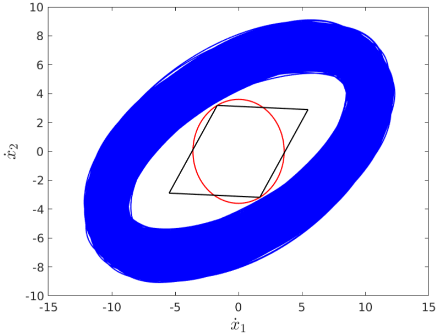
then $\bar{\bar{\mathcal{V}}}_x^{\mathcal{G}} \subseteq \mathcal{V}_x^{\mathcal{G}}$.

Corollary

For invertible matrices $G(x_0)$, $\bar{\mathcal{V}}_x^{\mathcal{G}} \subseteq \bar{\bar{\mathcal{V}}}_x^{\mathcal{G}}$.

Visual Interpretation of GVS Underapproximations





Interpretation of Ball ODI as a Control System

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  as a control system
- The reachable set of such a control system contained in the guaranteed reachable set

Theorem

Let us consider the control system

$$\dot{x} = a + g(\|x\|)\bar{u}, \quad x(0) = x_0,$$

on $\{x \mid \|x\| \le \|G^\dagger(x_0)\|/(L_f + L_G)\}$, with $a = f(x_0)$, $\bar{u} \in \mathbb{B}^n(0;1) \cap \mathrm{Im}(G(x_0))$ and where $g(s) = \|G^\dagger(s_0)\|^{-1} - (L_f + L_G)s$ if $s \le \|G^\dagger(s_0)\|^{-1}/(L_G + L_f)$. If $\overline{\mathcal{R}}(T, x_0)$ is the reachable set of the control system above, then $\overline{\mathcal{R}}(T, x_0) \subseteq \mathcal{R}^{\mathcal{G}}(T, x_0)$.

Interpretation of the Polygon ODI as a Control System

- We interpret the following ODI:

$$\dot{x} \in P(\mathcal{S}(x)), \quad x(0) = x_0,$$

as a control system.

Theorem

Let $s \in \mathbb{R}$ and $g(s) = \|G(s_0)^{\dagger}\|^{-1} - (L_G + L_f)s$, $\alpha(s) = \|G(s_0)^{\dagger}\|^{-1} - L_G s$, $\beta(s) = \mu\|G(s_0)^{\dagger}\|L_G s$ with μ as defined in Theorem 2. Let $U\Sigma V^T$ be the singular value decomposition of $G(x_0)$ where $U = [\eta_1, ..., \eta_n]$. Let $r = \mathrm{rank}(G(x_0))$; we define $\Lambda(s) \in \mathbb{R}^{n\times m}$ such that $\mathrm{diag}(\Lambda(s)) = \max\{\frac{g(s)}{\alpha(s)\|G(x_0)^{\dagger}\eta_i\|+\beta(s)}, g(s), 0\}$ and $\lambda_{ij}(s) = 0$ elsewhere. The reachable set of $\dot{x} \in P(\mathcal{S}(x))$ equals the reachable set of the control system

$$\dot{x} = a + U\Lambda(\|x\|)u, x(0) = x_0,$$

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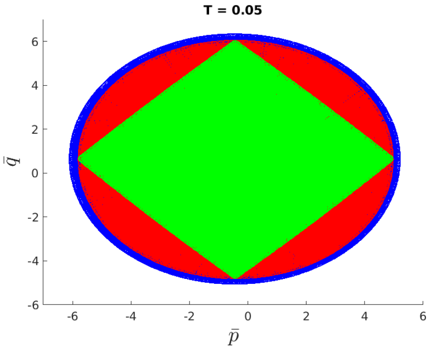
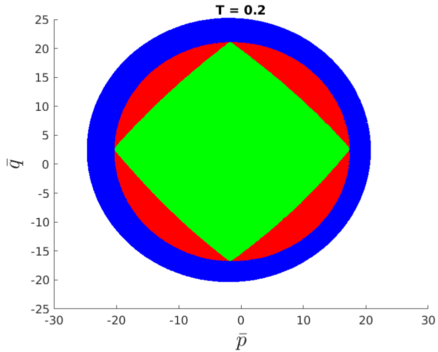
Visual Interpretation of the Results





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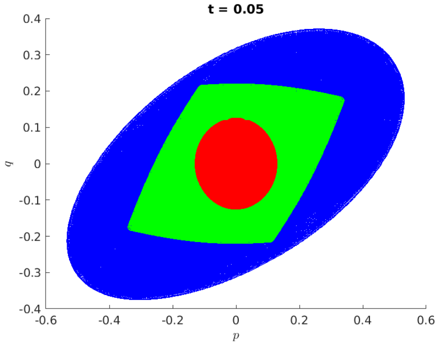

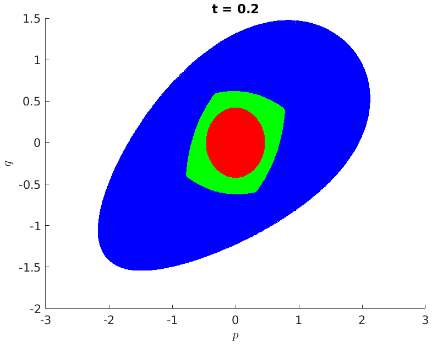

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- The problem statement remains consistent with previous work

New Method for Underapproximating the GVS

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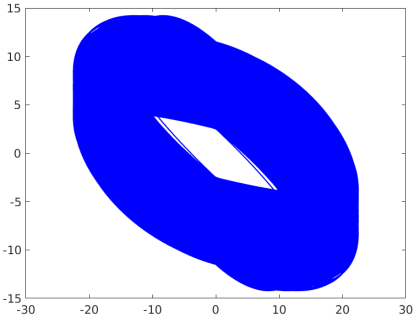

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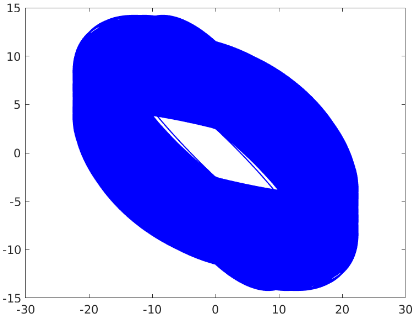

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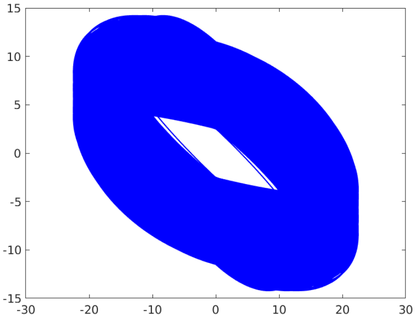

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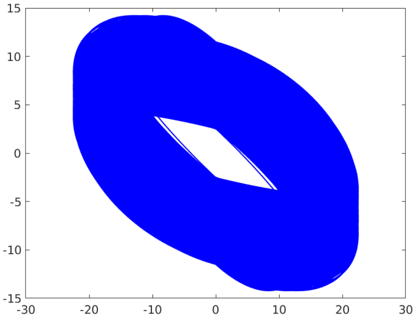


Finite Perturbation Theorem

Within a domain largely consistent with previous derivations, we can reduce the infinite constraint optimization problem to one with finitely many constraints

Finite Perturbation Theorem

- Let δ be the maximum perturbation magnitude for an element of $G(x_0)$
- If $\exists u_1, u_2 \in \mathcal{U}$ s.t.

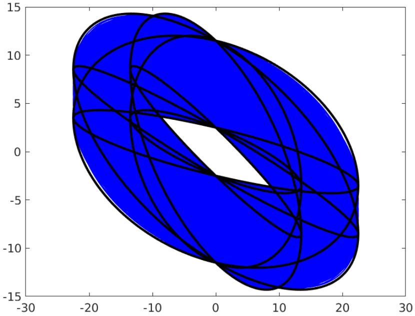
$$v = \left(G(x_0) + \begin{bmatrix} +\delta & 0 \\ 0 & 0 \end{bmatrix}\right) u_1,$$

$$v = \left(G(x_0) + \begin{bmatrix} -\delta & 0 \\ 0 & 0 \end{bmatrix}\right) u_2,$$

then $\forall\, \alpha \in [-1, 1]$, $\exists u^* \in \mathcal{U}$ s.t.

$$v = \left(G(x_0) + \begin{bmatrix} \alpha\delta & 0 \\ 0 & 0 \end{bmatrix}\right) u^*.$$

- Any vector contained in the 2^{n^2} black edges are also contained in the blue

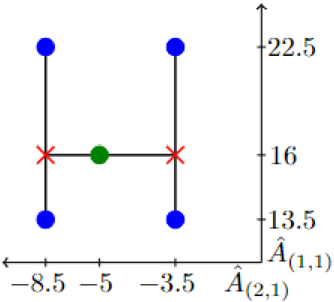


Finite Perturbation Theorem - Example

- Let $\Delta = \begin{bmatrix} 4.5 & 0 \\ 2.5 & 4.5 \end{bmatrix}$ be the matrix of maximal element-wise perturbation magnitudes
- Let $A = \begin{bmatrix} 18 & 0 \\ -6 & 7 \end{bmatrix}$
- Let $C_\Delta(A)$

$$= \left\{ \begin{bmatrix} 18 + \alpha_1 4.5 & 0 \\ -6 + \alpha_2 2.5 & 7 + \alpha_3 4.5 \end{bmatrix} \;\middle|\; \alpha_i \in [-1, 1] \right\}$$

- $\hat{A} = \begin{bmatrix} 16 & 0 \\ -5 & 5.5 \end{bmatrix} \in C_\Delta(A)$



Maximally Inscribed Ellipsoid

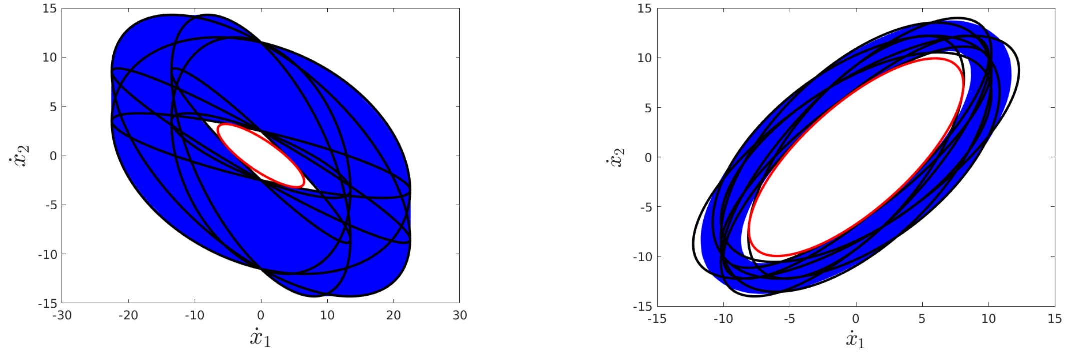
Theorem

Let A and Δ_k be the nominal and perturbation matrices respectively. Let $U\Sigma V^T$ be the singular value decomposition of A and let $\Sigma_k = U^T(A+\Delta_k)V$ and $A_k = (\Sigma_k^{-1})^T\Sigma_k^{-1}$. Let $\mathcal{E}_k = (A+\Delta_k)\mathcal{U}$. Then, an ellipsoid $\mathcal{E}$ of maximal volume such that $\mathcal{E} \subseteq \bigcap_k \mathcal{E}_k$ is given by $\mathcal{E} = UBV^T\mathcal{U}$ where B is the solution to

$$\begin{aligned} \underset{B\in\mathcal{S}^n_{++},\lambda_1,\ldots,\lambda_{2^{n^2}}\in\mathbb{R}}{\text{minimize}} \quad & \log\det B^{-1} \\ \text{subject to} \quad & \begin{bmatrix} -\lambda_k+1 & 0 & 0 \\ 0 & \lambda_k I_n & B \\ 0 & B & A_k^{-1} \end{bmatrix} \geq 0 \\ & \text{for all } k \in [2^{n^2}]. \end{aligned}$$

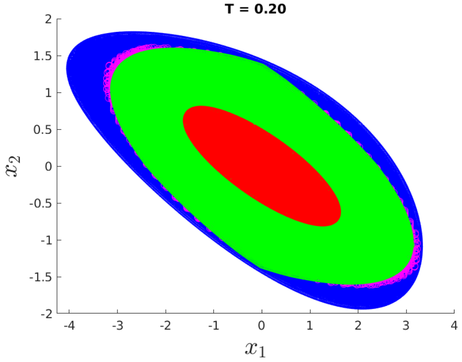
- The 2^{n^2} constraints can potentially be reduced with further analysis

Illustrated Results



Analysis of the Results

- We are able to characterize the GVS as an intersection of finitely many ellipsoids



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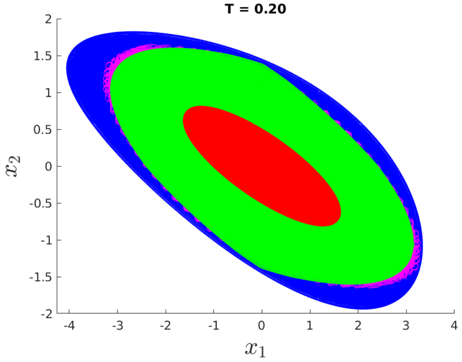

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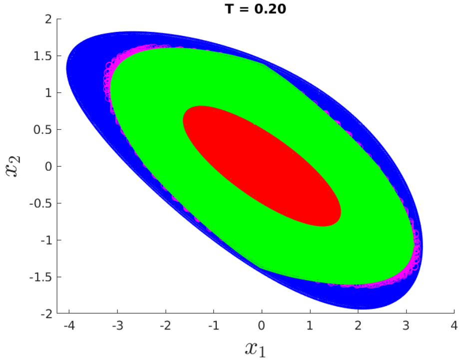

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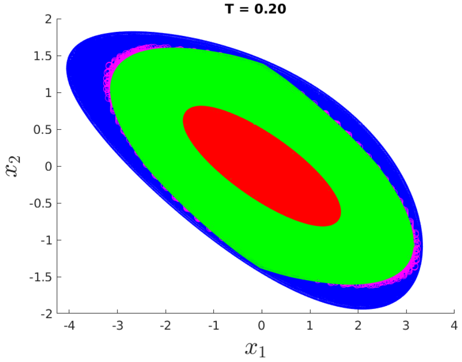
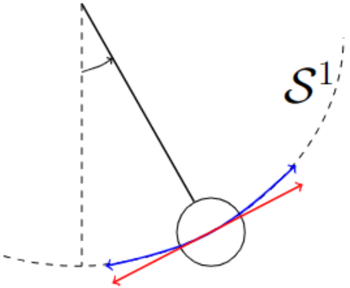
- We can take the worst-case perturbation for all x within a domain and solve one optimization problem
  - Real time implementable

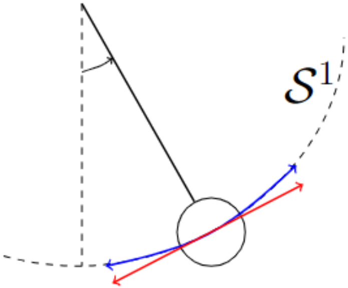


- Control systems can exist on manifolds outside of $\mathbb{R}^n$



Generalization of Results to Complete Manifolds

- Control systems can exist on manifolds outside of $\mathbb{R}^n$
  - Pendulum



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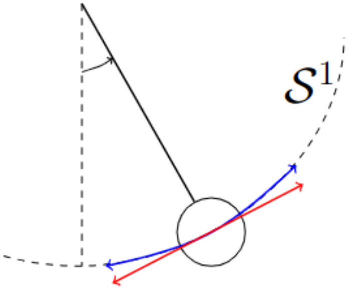

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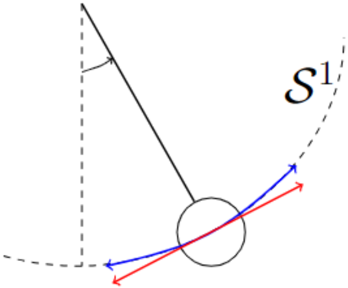

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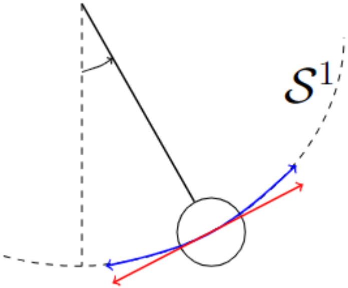

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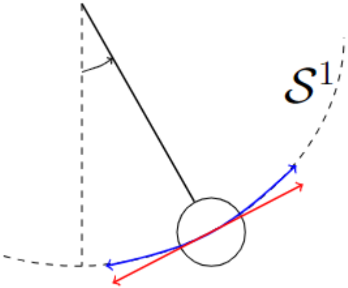
- We relax assumptions to assume the control system exists on a complete Riemannian manifold



Formal Problem Statement

- We have an unknown nonlinear control-affine system operating on a manifold M of the form:
  - $\dot{x} = f(x) + G(x)\mathcal{U} = f(x) + \sum_l g_l(x)u^l, \quad x(0) = x_0$
  - $f(x) \in M$ and $g_l(x) \in M$
- We assume knowledge of:
  - The initial state x_0
  - The input set $\mathcal{U} = \mathbb{B}^m(0;1)$
  - Local dynamics $f(x_0)$ and $G(x_0)$
    - Learned within an arbitrarily small error from test control inputs
  - The maximum growth rate of dynamics given by *Riemannian Lipschitz* bounds L_f and L_G
    - Determined from known physical laws
    - Uncertainty quantification

Problem Statement

Determine or underapproximate the guaranteed reachable set

Preliminaries for Control Systems on Manifolds

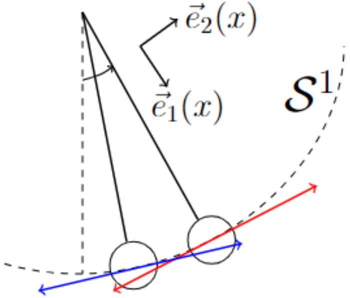
- Operations between vector spaces require connections in the geometric sense

Riemannian Lipschitz

Let V be a continuous vector field on M and τ be the parallel transport. Then L is the *classical Lipschitz constant* on V if

$$L = \sup_{\gamma} \frac{|\tau_\gamma V(\gamma(0)) - V(\gamma(1))|_{h_x}}{\mathrm{Length}(\gamma)}$$

where $\gamma : [0, 1] \to M$ varies over all $\mathcal{C}^1$-paths and τ_γ is shorthand for the parallel transport along the curve γ from $\gamma(0)$ to $\gamma(1)$.



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Lemma

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- Given an appropriate neighborhood, we need only consider one geodesic path
  - Knowledge of Lipschitz constant for general manifolds as reasonable as in the Euclidean case

Guaranteed Underapproximation of GVS

Riemannian Ball Underapproximation

Let $f(x_0)$, $G(x_0)$, L_f, L_G, H_x, Γ_{ij}^k, and g_l^Γ for $l \in [m]$ be defined as above. Let $\gamma : [0,1] \to M$ define a geodesic curve from x_0 to x. Let $\tilde{\tau}$ define the parallel transport using the flat connection. If

$$\overline{\mathcal{V}}_x^{\mathcal{G}} = \mathbb{B}^n \left(\tilde{\tau}_{x_0}^x f(x_0); \alpha(x_0, x)\right) \cap \mathrm{Im}(\tilde{\tau}_{x_0}^x G(x_0))$$

where $\overline{\mathcal{V}}_x^{\mathcal{G}} \in T_x M$, and

$$\begin{gathered} \alpha(x_0, x) = \|\tilde{\tau}_{x_0}^x G^\dagger(x_0)\|^{-1} - \\ (\|H_x^{-1}\|\|H_x\|)^{\frac{1}{2}} \Big(\|H_x\|^{\frac{1}{2}} \left\| \begin{bmatrix} g_1^\Gamma & \dots & g_m^\Gamma \end{bmatrix} \right\| + \\ \left\| \sum_{i,j,k} \dot{\gamma}^i \Gamma_{ij}^k f^j(x_0) \vec{e}_k \right\| + \left(L_g + \|H_x\|^{-\frac{1}{2}} L_f \right) d(x_0, x) \Big), \end{gathered} \tag{1}$$

then $\overline{\mathcal{V}}_x^{\mathcal{G}} \subseteq \mathcal{V}_x^{\mathcal{G}}$.

Calculate the GRS on a Manifold

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$$\dot{x} \in \overline{\mathcal{V}}_x^{\mathcal{G}}$$
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Theorem

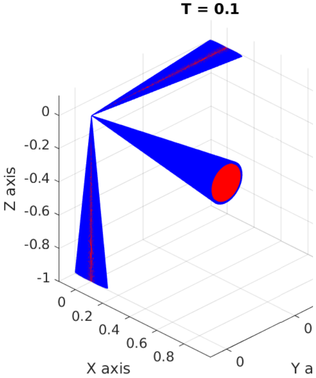
Let $\overline{\mathcal{R}}(T, x_0)$ be defined as the reachable set of

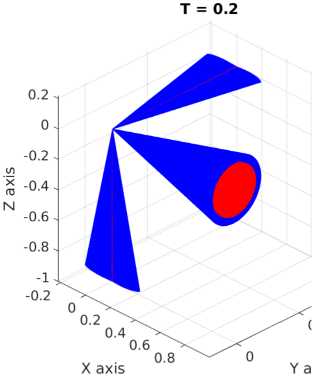
$$\dot{x} = a + g(x_0, x)u, \quad x(0) = x_0,$$

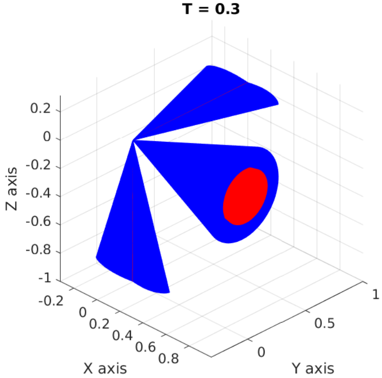
on $\{x \mid d(x_0, x) \leq \overline{d}(x_0, x)\}$, with $a = \tilde{\tau}_{x_0}^{x} f(x_0)$, $u \in \mathbb{B}^n(0; 1)$, and $g(s_0, s) = \alpha(x_0, x)$ if $d(s_0, s) \leq \overline{d}(x_0, x)$. Then $\overline{\mathcal{R}}(T, x_0) \subset \mathcal{R}^{\mathcal{G}}(T, x_0)$.

- The domain $\overline{d}(x_0, x)$ is explicitly defined in the literature

Illustrated Results







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- We now want to determine how we can use this information to synthesize control action for unknown nonlinear systems

System Identification

- Create a mapping from input trajectories to observed output trajectories
  - Recursive least squares approach
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  - Recursive least squares approach
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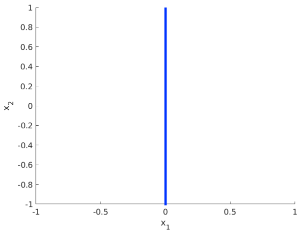
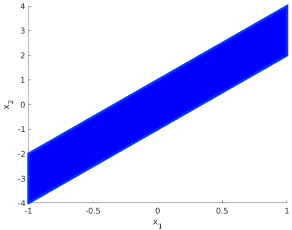
- Requires knowledge of individual trajectories and control of actuators
- What if the only information one had access to is the reachable sets?

Reachable Sets

- We have the system dynamics
  - $x[i+1] = Ax[i] + bu[i], \quad x[0] = 0$
- For $i \in \mathbb{Z}_{\geq 0}$, the (forward) reachable set for the system at time i is
  - $\mathcal{R}(i, x[0]) = \{\phi_u(i; x[0]) \mid u : \mathbb{Z}_{\geq 0} \to \mathcal{U}\}$
- Defines the set of states which are reachable at time i for the system using all $u \in \mathcal{U}$
- Calculated through Minkowski sums

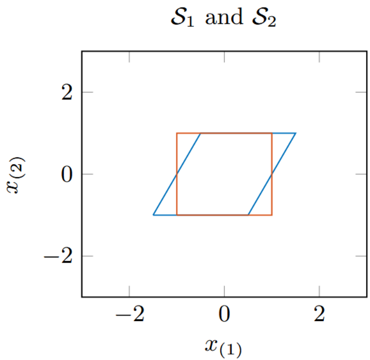


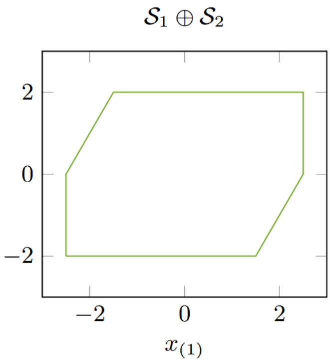


Minkowski Sum

- Given two sets $\mathcal{S}_1, \mathcal{S}_2 \in \mathbb{R}^n$ we denote

$$\mathcal{S}_1 \oplus \mathcal{S}_2 = \{s_1 + s_2 \mid s_1 \in \mathcal{S}_1,\ s_2 \in \mathcal{S}_2\}$$





Technical Assumptions

- Uniquely determine system model under the following assumptions:
  - Single-Input System
  - Generic Properties
  - Discrete Linear System
    - $x[i+1] = Ax[i] + bu[i], \quad x[0] = 0, \quad u \in \mathcal{U}$
    - Consecutive unit-length time steps
  - Fully Controllable System
  - The input set $\mathcal{U}$ is completely known

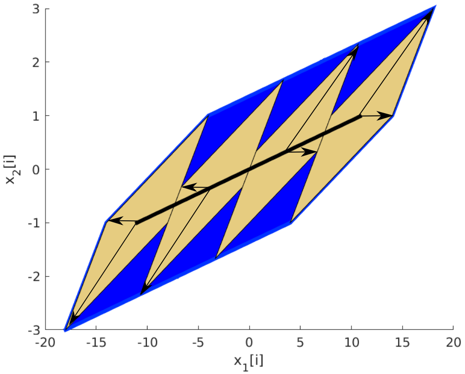
How can we solve for A and b given the assumptions above?

Reachable Sets as Minkowski Sums

- We have the following system:
  - $x[i+1] = Ax[i] + bu[i], \quad x[0] = 0, \quad u \in \mathcal{U}$
- $\mathcal{R}(i,0) =$ the reachable set of the system above at time i
- $\mathcal{R}(1,0) = b\mathcal{U}$
- $x[i] = A^i x[0] + A^{i-1} bu[0] + \ldots + bu[i-1]$
- This implies:
  - $\mathcal{R}(i,0) = A^{i-1} b\mathcal{U} \oplus \ldots \oplus b\mathcal{U}$
  - $\mathcal{R}(i,0) = A^{i-1} b\mathcal{U} \oplus \mathcal{R}(i-1,0)$
- Therefore:
  - $A^{i-1} b\mathcal{U} = \mathcal{R}(i,0) \ominus \mathcal{R}(i-1,0)$

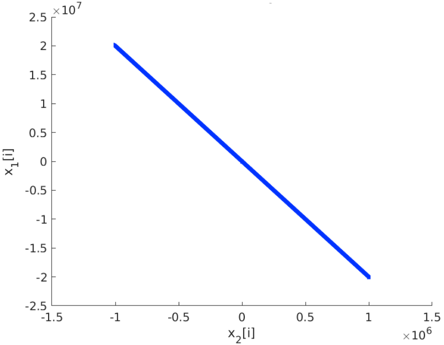
Minkowski Difference

- Given two sets $\mathcal{A}, \mathcal{B} \in \mathbb{R}^n$
  - $\mathcal{A} \ominus \mathcal{B} = \{c \in \mathbb{R}^n \mid c \oplus \mathcal{B} \subseteq \mathcal{A}\}$
- Let $v^{(i)} \in \mathcal{V}$ be the vertices of $\mathcal{R}(i-1,0)$
  - $\mathcal{R}(i,0) \ominus \mathcal{R}(i-1,0) = \bigcap_{v^{(i)} \in \mathcal{V}} (\mathcal{R}(i,0) - v^{(i)})$



Solving for Dynamics

- Solving for the vector b with knowledge of $\mathcal{U}$ is trivial
  - $\mathcal{R}(1,0) = b\mathcal{U}$
- Similarly we can solve for $A^i b$ with knowledge of $\mathcal{U}$
  - $A^{i-1}b\mathcal{U} = \mathcal{R}(i,0) \ominus \mathcal{R}(i-1,0)$
- If we have a controllable system
  - $C_{A,b} = \begin{bmatrix} b & Ab & \dots & A^{i-1}b \end{bmatrix}$
  - $C_{A,b}$ is invertible
  - $A = AC_{A,b}C_{A,b}^{-1}$



Uniqueness

Can we determine if the calculated dynamics A and b are unique?

- In general cases no
  - $A = I, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
  - $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

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  - Any generic system with an input set $\mathcal{U}$ asymmetric around the origin
  - Generic 2D system with an input set $\mathcal{U}$ symmetric around the origin

Analytical Results for Uniqueness

Theorem

Under generic assumptions, we can **uniquely** identify the unknown dynamics of a discrete linear system with an input set asymmetric around the origin using $n + 1$ reachable sets for unit time intervals

Remark

The theorem above holds for systems with symmetric input sets of dimension 2

Bandpass Filter Circuit Example

- Fourth-order band-pass circuit
  - Controllable canonical representation

$$x[i+1] = Ax[i] + bv_c[i] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} x[i] + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v_c[i]$$

- $v_c[i] \in [0,1]$ for all $i \in \mathbb{Z}_{\geq 0}$
- If $a_0 \neq 0$ then assumptions are likely satisfied
  - A invertible
  - A diagonalizable
- Want to recover the true parameters
  - $a_0 = 3,\ a_1 = 2,\ a_2 = 3,\ a_3 = 6$

Bandpass Filter Circuit Example

$$\mathcal{R}(1,0) = \mathrm{conv}\left(\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right), \mathcal{R}(2,0) = \mathrm{conv}\left(\begin{bmatrix}0\\0\\1\\-5\end{bmatrix}, \begin{bmatrix}0\\0\\1\\-6\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)$$

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$$\mathcal{R}(4,0) = \mathrm{conv}\left(\begin{bmatrix}-0.15\\1.05\\-5.99\\33.00\end{bmatrix}, \begin{bmatrix}0.85\\-5.95\\34.01\\-188\end{bmatrix}, \begin{bmatrix}0.85\\-5.95\\34.01\\-187\end{bmatrix}, \begin{bmatrix}-0.15\\1.05\\-5.99\\34.00\end{bmatrix}\right)$$

$$\mathcal{R}(5,0) = \mathrm{conv}\left(\begin{bmatrix}-5.93\\33.88\\-188.02\\1035.00\end{bmatrix}, \begin{bmatrix}1.07\\-6.12\\33.98\\-188.00\end{bmatrix}, \begin{bmatrix}1.07\\-6.12\\33.98\\-187.00\end{bmatrix}, \begin{bmatrix}-5.93\\33.88\\-188.02\\1036.00\end{bmatrix}\right)$$

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- If we want to find A for an n dimensional system, we can do so with $n+1$ reachable sets

Further Analysis

- To apply this method to our previous results, we need to generalize these results to the continuous domain for multi-input systems
  - Can potentially avoid generalizing to nonlinear systems for short-time intervals
  - Zonotopes are not closed under Minkowski difference when $\mathcal{U} \subseteq \mathbb{R}^m$ and $m > 2$

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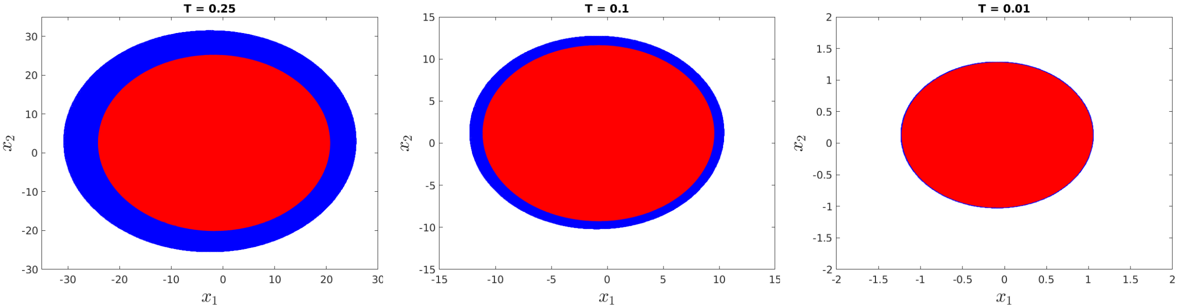
Reachable Sets as $t \to 0$



Review of Original Assumptions with New Problem Statement

- We have an unknown nonlinear control-affine system of the form:
  - $\dot{x} = f(x) + G(x)\mathcal{U}, \quad x(0) = x_0$
  - $f(x) \in \mathbb{R}^n$ and $G(x) \in \mathbb{R}^{n \times m}$
- We assume knowledge of:
  - The initial state x_0
  - The input set $\mathcal{U} = \mathbb{B}^m(0; 1)$
  - Local dynamics $f(x_0)$ and $G(x_0)$
    - Learned within an arbitrarily small error from test control inputs
  - The maximum growth rate of dynamics given by Lipschitz bounds L_f and L_G
    - Determined from known physical laws
    - Uncertainty quantification

Problem Statement

Synthesize control action for an unknown nonlinear system

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Proposition

Suppose $x_0 \in \mathrm{Int}(\mathbb{B})$ and $y \in \partial\hat{\mathcal{R}}(T, x_0)$. Then, $\hat{u} \equiv \frac{y - aT - x_0}{|y - aT - x_0|}$ is the unique control (almost everywhere) such that $\hat{\phi}_{\hat{u}}(T) = y$. Consequently, there exists a unique controlled path from x_0 to y.

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- We can determine the GRS at some state, reach some $y \in \partial\hat{\mathcal{R}}(T, x_0)$, then repeat the process to follow some piecewise linear path and reach some eventual desired end state

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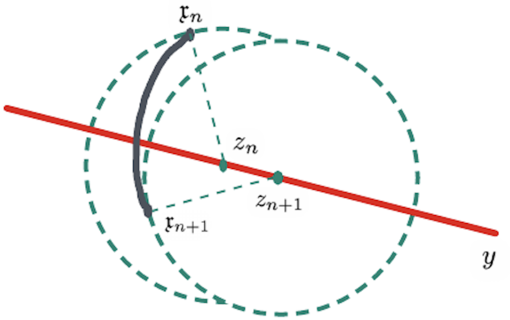

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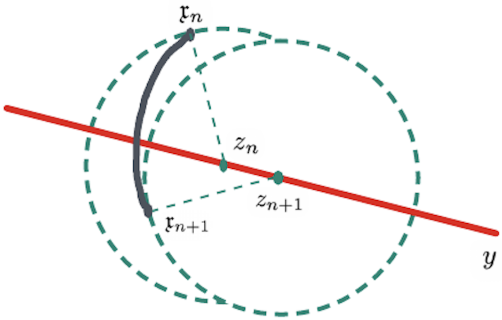

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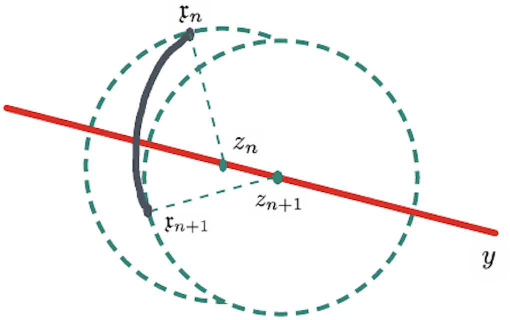

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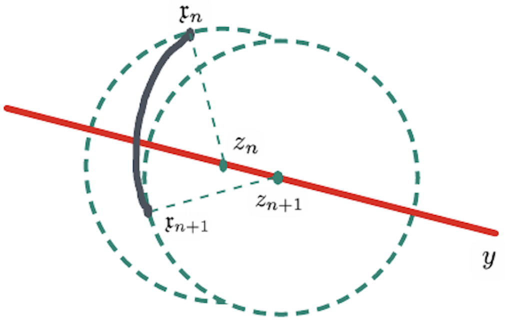

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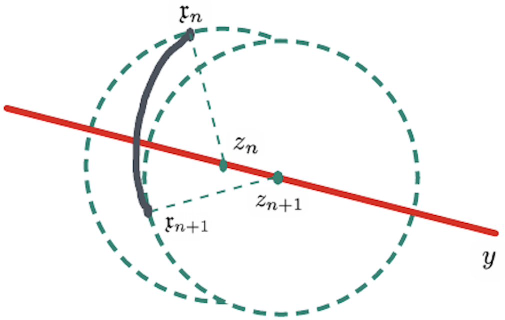
  - Each input u_n forms a direct path towards z_n with an arbitrarily small error
    - As $z_n \to y$, so to does the control system



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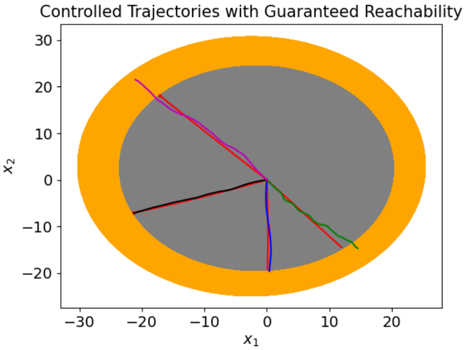

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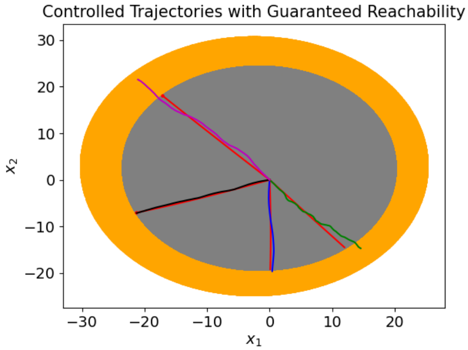

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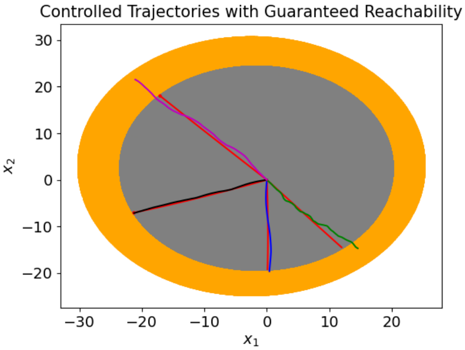

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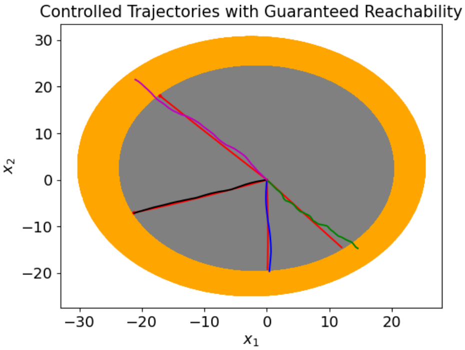

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  - Demonstrate RPC effectively on a system without knowledge of the system dynamics
- The ultimate goal is to illustrate by example how RPC can control an unknown nonlinear system and analyze its efficacy

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- We use the notion of *eventual reachabilty*

Resilient Task Assignment for Underactuated Systems

Eventual (Forward) Reachability

We define the eventual forward reachable set of $\mathcal{M}(f, G)$ as

$$\mathcal{R}^{f,G}(T, x_0) = \left\{ \bigcup_{t \in [0,T]} \phi_u^{f,G}(t; x_0) \mid u : [0, T] \to \mathcal{U},\ t \in [0, T] \right\}$$

where $\mathcal{R}^{f,G}(T, x_0)$ represents the set of states that can be reached within some time T.

- For short time intervals the exact amount of time it takes to reach a state is less significant
  - Interested in determining if a state is provably reachable
  - Not important to know the exact time that state is reached

Resilient Task Assignment

Lemma

Let

$$\dot{x} = f(x(t)), \quad x(0) = x_0 \tag{2}$$

and let $\lambda : \mathbb{R}^n \to [1, +\infty)$ such that

$$\dot{z} = \lambda(z(t)) f(z(t)), \quad z(0) = z_0. \tag{3}$$

If $x_0 = z_0$ and $\lim_{t\to\infty} \int_0^t \frac{1}{\lambda(x(s))}\, ds = \infty$, then there exists $t \in [0, T]$ such that $z(t) = x(T)$.

- If two velocities are colinear and the initial state $x(0) = z(0)$, then any state reached by (2) at some time T would have been reached by (3) for some time $t \in [0, T]$

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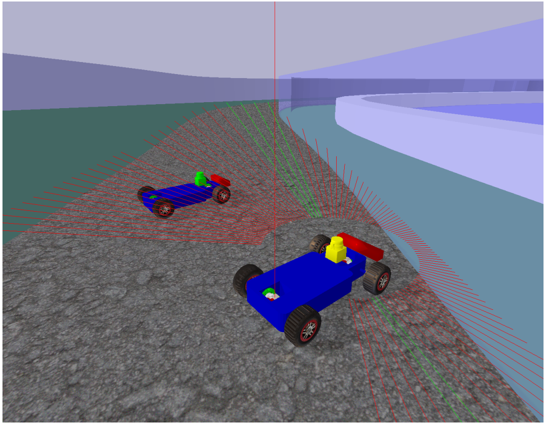

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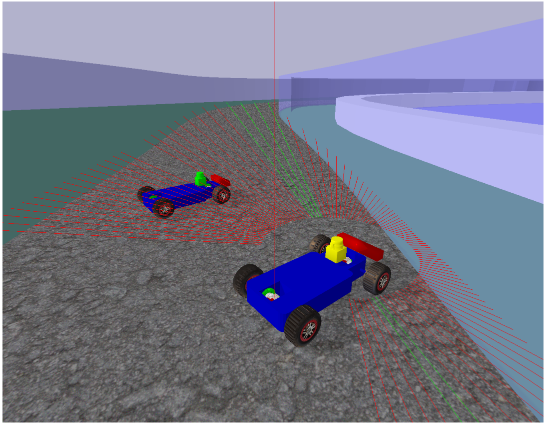

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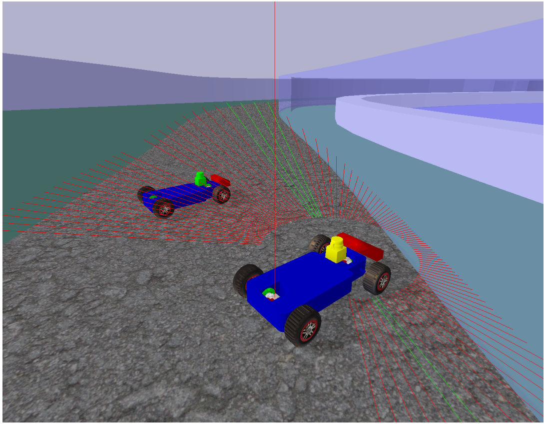

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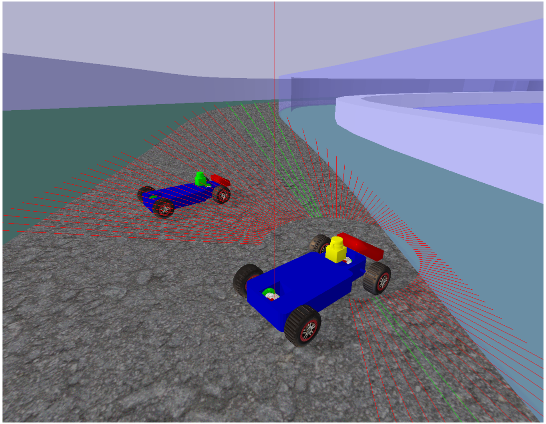

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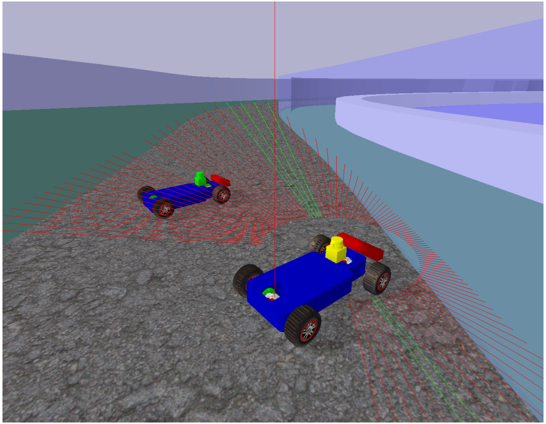

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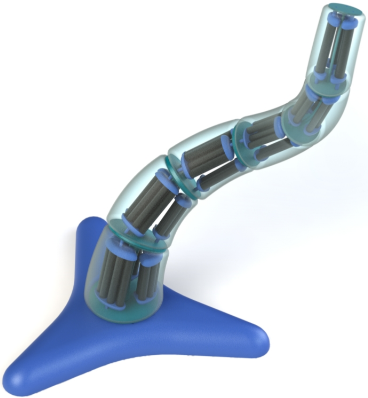
- Use information from machine learning to gain knowledge consistent with assumptions outlined in previous work for resilient task assignment
- Apply RPC to accomplish a provably achievable performance objective



Control of Unknown Systems Using the Koopman Operator

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Koopman Operator Family

The Koopman operator family $\{\mathcal{K}\}_{t\geq 0}$ of the nonlinear system is a collection of maps $\mathcal{K}_t : \mathcal{F} \to \mathcal{F}$ defined by

$$\mathcal{K}_t h = h \circ \phi(t, \cdot), \quad h \in \mathcal{F}$$

where $\circ$ denotes the composition of ϕ with $\mathcal{F}$.

Control of Unknown Systems Using the Koopman Operator

- For autonomous systems

$$\dot{x} = f(x),$$

  there is a semigroup of Koopman operators $\mathcal{K}_{\Delta t}$ associated with the flow map for time interval Δt

  - The infinitesimal generator for this semigroup is defined as

$$\mathcal{L}h(x) := \lim_{t \to 0} \frac{\mathcal{K}_t h(x) - h(x)}{t} \approx \int_0^{\tau} e^{-\lambda s} h(\phi(s, x)) \, ds$$

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with any fixed $\tau > 0$ for a sufficiently large λ
- Define $p_j : \mathcal{X} \to \mathbb{R}$ as the projection function to the j^{th} dimension
- We can learn f_j at each j at samples points $\{x^m\}$ with
$$f_j(x^{(m)}) \approx \mathcal{L}p_j(x) = \int_0^{\tau} e^{-\lambda s} p_j(\phi(s,x))\, ds$$

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$$\dot{x} = f(x) + G(x)u = f(x) + \sum_{l} g_l(x)u^l, \quad x(0) = x_0$$

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- With enough data, we can perform system identification to learn f and g_l within some error bound
  - Additionally, we can determine long-term stability guarantees
- Further improvement involves adaptation of onboard capabilities
  - Determining how information gathered onboard can improve error bounds and performance

Publications

[1] T. Shafa and M. Ornik, "Reachability of nonlinear systems with unknown dynamics," *IEEE Transactions on Automatic Control*, vol. 68, no. 4, pp. 2407–2414, 2022.

[2] ——, "Maximal ellipsoid method for guaranteed reachability of unknown fully actuated systems," in *2022 IEEE 61st Conference on Decision and Control (CDC)*. IEEE, 2022, pp. 5002–5007.

[3] ——, "Guaranteed reachability on riemannian manifolds for unknown nonlinear systems," *arXiv preprint arXiv:2404.09850*, 2024.

[4] T. Shafa, R. Dong, and M. Ornik, "Identifying single-input linear system dynamics from reachable sets," in *2023 62nd IEEE Conference on Decision and Control (CDC)*. IEEE, 2023, pp. 3527–3532.

[5] Y. Meng, T. Shafa, J. Wei, and M. Ornik, "Online learning and control synthesis for reachable paths of unknown nonlinear systems," *arXiv preprint arXiv:2403.03413*, 2024.

Questions

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