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On the Minimax Reachability of Target Sets and Target Tubes*

Sur la capacité minimax d'atteindre des séries de buts et des enveloppes de buts

Über die Minimax-Erreichbarkeit von Zielmengen und Ziel-"Schläuchen"

О минимаксной способности достижения рядов целей и оболочек целей

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Feedback control of uncertain dynamic systems may be derived such that the system state is guaranteed to lie in a specified region of state space in the presence of input and observation disturbances.

Summary—This paper is concerned with the closed-loop control of discrete-time systems in the presence of uncertainty. The uncertainty may arise as disturbances in the system dynamics, disturbances corrupting the output measurements or incomplete knowledge of the initial state of the system. In all cases, the uncertain quantities are assumed unknown except that they lie in given sets. Attention is first given to the problem of driving the system state at the final time into a prescribed target set under the worst possible combination of disturbances. This is then extended to the problem of keeping the entire state trajectory in a given target "tube". Necessary and sufficient conditions for reachability of a target set and a target tube are given in the case where the system state can be measured exactly, while sufficient conditions for reachability are given for the case when only disturbance corrupted output measurements are available. An algorithm is given for the efficient construction of ellipsoidal approximations to the sets involved, and it is shown that this algorithm leads to linear control laws. The application of the results in this paper to pursuit-evasion games is also discussed.

1. INTRODUCTION

TWO BASIC problems of deterministic control theory are the controllability problem and the tracking (servomechanism) problem. The controllability problem is concerned with transferring the state of a system from an initial state-time pair to a final state-time pair. The tracking problem is concerned with keeping the state-trajectory of the system "sufficiently close" to a prescribed target trajectory.

In this paper we examine two analogs of these problems that arise when there is uncertainty about the system state. This uncertainty can arise because the initial state of the system is not known exactly and because the system dynamics and output measurements are corrupted by "noise". The most commonly used approach in such situations is to model the initial state as a random vector and the dynamics and measurement noises as additive stochastic processes. Under these circumstances, a possible analog to the controllability problem is to reach a target *set* at the final time with a prescribed probability or degree of certainty, while the usually-adopted analog of the tracking problem is to take an "on-the-average" approach and minimize the expectation of a cost functional that depends quadratically on the deviation between the system trajectory and the target trajectory. If the system is linear, the initial state is a Gaussian random vector, the system and measurement noises are independent white Gaussian processes, and the cost functional also depends quadratically on the control, the solution to this latter problem is given by the well-known "separation theorem" or "certainty equivalence principle".

The approach adopted in this paper differs from those outlined above in two ways. First, the uncertainties are not modelled as random vectors or stochastic processes but instead are considered unknown except for the fact that they belong to prescribed, bounded sets. Secondly we adopt a pessimistic "worst case" or "guaranteed performance" approach rather than the usual "on the average" approach. Thus we seek the controller that achieves the desired objective or performs "best" under the worst possible combination of disturbances. Under these conditions, the most natural analog of the deterministic controllability problem is that of steering the system state into a

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desired final target set under all possible combinations of disturbances. In other words, we would like to design a controller in such a way as to guarantee that the final state of the system will always lie in a prescribed target set despite the presence of uncertainties. In a similar vein, a natural analog of the tracking problem under these same conditions is to keep the entire state trajectory in a "tube" containing the desired trajectory under all possible disturbances. We refer to these two problems as those of "reachability of a target set" and "reachability of a target tube". Possible applications of these two problems can be expected in the control of systems under uncertainty where either a set description of the uncertain quantities is more readily available than a probabilistic one, or where specified tolerances must be met with certainty. Such applications can be found in diverse areas such as in chemical process control cases where the state must stay in a specified region of the state space, or equivalently avoid a critical region, in aerospace applications such as a spacecraft reentry problem, etc.

The modelling of uncertainties and disturbances as quantities that are unknown except that they belong to prescribed sets and the adoption of a "worst case" or "guaranteed performance" viewpoint have both received attention before. The state estimation problem under these circumstances has recently been discussed in Refs. [1–5]. The reachability of target sets and target tubes with open-loop controls has been discussed in Ref. [6], and certain aspects of the problem of the reachability of a target set by a closed loop controller using perfect measurements of the system state have been discussed in Refs. [1] and [7] in the framework of a more general problem. In this paper we examine the reachability of target sets and target tubes by *closed-loop controllers* utilizing either perfect measurements of the system state or imperfect measurements of the system output. In order to achieve the greatest transparency of the ideas involved, we concentrate our attention on discrete-time dynamic systems.

In section 2 we consider the reachability of a target set by the state of a non-linear discrete dynamic system using perfect measurements of the state. We give a geometric necessary and sufficient condition for existence of control laws that achieve reachability, and characterize these control laws. These results are extended in section 3 to the case of reachability of a target tube. In section 4 we consider the special case of a linear system and obtain some additional results and characterizations. We also give a polyhedral algorithm and an ellipsoidal approximation algorithm for construction of sets required for the solution and show that the ellipsoidal algorithm provides linear control laws. Section 5 contains a discussion of some relationships between the reachability problem and the "unknown but bounded" state estimation problem that has been examined in Refs. [1–5]. In section 6 we consider the reachability problems for the case where the controller has available only imperfect measurements of the system output and give some sufficient conditions for reachability that make use of an estimator derived by the authors [5]. Finally in section 7 we point out applications of our results to pursuit evasion games.

2. REACHABILITY OF A TARGET SET

In this section we examine the reachability of a target set when perfect measurements of the system state are available to the controller at each time.

Problem 1. Consider the discrete-time dynamic system

$$\mathbf{x}_{k+1}=\mathbf{f}_k(\mathbf{x}_k,\ \mathbf{u}_k)+\mathbf{g}_k(\mathbf{w}_k) \tag{1}$$

where $\mathbf{x}_k \in X=R^n$ (*n*-dimensional Euclidian space) is the state vector, the control vector $\mathbf{u}_k$ is to be selected from a prescribed set $\mathbf{U}_k \subset R^r$, the disturbance vector $\mathbf{w}_k$ is assumed to belong to a given bounded set $\mathbf{W}_k \subset R^p$, the initial state $\mathbf{x}_o$ is assumed to be contained in a given set $\mathbf{X}_o \subset R^n$, and the functions $\mathbf{f}_k$: $R^n \times R^r \to R^r$ and $\mathbf{g}_k$: $R^n \to R^n$ are known. Given a prescribed target set $\mathbf{X}_N \subset R^{n*}$, find, if it exists, a closed-loop control law $\mathbf{u}(\cdot,\ \cdot)$ mapping the pairs $(\mathbf{x}_k, k)$ into $\mathbf{U}_k$, $k=0,\ 1 \ldots, N-1$, with the property that at time N the state $\mathbf{x}_N$ of the closed-loop system

$$\mathbf{x}_{k+1}=\mathbf{f}_k(\mathbf{x}_k,\ \mathbf{u}(\mathbf{x}_k,\ k))+\mathbf{g}_k(\mathbf{w}_k) \tag{2}$$

is contained in $\mathbf{X}_N$ for all possible disturbance sequences $\mathbf{w}_k \in \mathbf{W}_k$, $k=0,\ 1, \ldots, N-1$, and all possible initial states $\mathbf{x}_0 \in \mathbf{X}_0$.

Definition 1. The target set $\mathbf{X}_N$ is reachable at time N from the initial state set $\mathbf{X}_0$ at time 0 if there exists at least one solution to Problem 1.

We remark that if we are to guarantee reachability of the target set under all possible disturbances, we must take the pessimistic viewpoint of attributing to "Nature" the role of an active adversary who selects the disturbances at each time in such a way as to try to prevent the system from reaching the target set. Thus we may view the

* If an output target set $\mathbf{Y}_N$ is to be reached where the output is $\mathbf{y}_k=\mathbf{h}_k\ (\mathbf{x}_k)$ we define the state target set

$$\mathbf{X}_N=\{\mathbf{x}_N:\ \mathbf{y}_N=\mathbf{h}_N(\mathbf{x}_N),\ \mathbf{y}\in\mathbf{Y}_N\}$$

and reduce the problem to the same form as above.

reachability problem as one in which there is the following sequence of $2N+1$ "moves" alternating between the Controller and Nature, each move being made with full knowledge of the outcomes of prior selections: (1) Nature selects $\mathbf{x}_o$ (2) Controller selects $\mathbf{u}_o$ (3) Nature selects $\mathbf{w}_o$ (4) Controller selects $\mathbf{u}_1, \ldots$ (2N) Controller selects $\mathbf{u}_{N-1}$, (2N+1) Nature selects $\mathbf{w}_{N-1}$. One can, in fact, convert this problem to a sequential minimax control problem defining by a cost function J to be the characteristic function of the complement of the set $\mathbf{X}_N$, i.e.

$$J(\mathbf{x}_N)=\begin{cases}0 & \mathbf{x}_N\in\mathbf{X}_N\\ 1 & \mathbf{x}_N\notin\mathbf{X}_N\end{cases}$$

where the Controller attempts to minimize J and Nature attempts to maximize it. This minimax control problem can be solved by dynamic programming [7]; in fact, the results of this section constitute a geometric solution to the dynamic programming algorithm, although we prefer to argue directly rather than view the problem as one of finding min max J.

In order for $\mathbf{x}_N=\mathbf{f}_{N-1}(\mathbf{x}_{N-1}, \mathbf{u}_{N-1})+\mathbf{g}_{N-1}(\mathbf{w}_{N-1})$ to be an element of $\mathbf{X}_N$ for all $\mathbf{w}_{N-1}\in\mathbf{W}_{N-1}$ it is clearly and trivially both necessary and sufficient that $\mathbf{f}_{N-1}(\mathbf{x}_{N-1}, \mathbf{u}_{N-1})$ belong to the *effective target set* $\mathbf{E}_N$ defined by

$$\mathbf{E}_N=\{\mathbf{z}\in R^n\colon\ [\mathbf{z}+\mathbf{g}_{N-1}(\mathbf{w}_{N-1})]\in\mathbf{X}_N,\ \forall\mathbf{w}_{N-1}\in\mathbf{W}_{N-1}\} \tag{3}$$

which, in turn, will occur for some $\mathbf{u}_{N-1}\in\mathbf{U}_{N-1}$ if and only if $\mathbf{x}_{N-1}$ is an element of the *updated target set* $\mathbf{T}_{N-1}$ defined by

$$\mathbf{T}_{N-1}=\{\mathbf{z}\in R^n\colon\ \mathbf{f}_{N-1}(\mathbf{z}, \mathbf{u}_{N-1})\in\mathbf{E}_N \text{ for some } \mathbf{u}_{N-1}\in\mathbf{U}_{N-1}\}. \tag{4}$$

Thus, as a direct consequence of the definitions of $\mathbf{E}_N$ and $\mathbf{T}_{N-1}$ in (3) and (4), we have the following three equivalent statements:

(1) $\mathbf{x}_N\in\mathbf{X}_N$ for all $\mathbf{w}_{N-1}\in\mathbf{W}_{N-1}$ and some $\mathbf{u}_{N-1}\in U_{N-1}$ if and only if $\mathbf{x}_{N-1}\in\mathbf{T}_{N-1}$, where $\mathbf{x}_{N-1}$ is the state at time $N-1$.

(2) $\mathbf{X}_N$ is reachable at time N from all points of a given set $\mathbf{X}_{N-1}$ of states $\mathbf{x}_{N-1}$ at time $N-1$ if and only if $\mathbf{X}_{N-1}\subset\mathbf{T}_{N-1}$.

(3) $\mathbf{X}_N$ is reachable at time N from the set $\mathbf{X}_o$ at time 0 if and only if $\mathbf{T}_{N-1}$ is reachable at time $N-1$ from $\mathbf{X}_o$ at time 0.

It should be noted that the set $\mathbf{E}_N$ and consequently the set $\mathbf{T}_{N-1}$ may be empty in which case the problem does not have a solution, i.e. the target set $\mathbf{X}_N$ is not reachable from any state $\mathbf{x}_{N-1}$ at time $N-1$ and hence from the initial condition set $\mathbf{X}_0$.

The reachability problem from time 0 to time N has thus been reduced to a reachability problem from time 0 to time $N-1$. Repeated application of the same procedure leads to a complete solution of the problem of reachability of a target set. To this end, define recursively the effective target set $\mathbf{E}_{k+1}$ at the time $k+1$ and the updated target set $\mathbf{T}_k$ at time k by [c.f. equations (3) and (4)]

$$\mathbf{E}_{k+1}=\{z\in R^n\colon\ [\mathbf{z}+\mathbf{g}_k(\mathbf{w}_k)]\in\mathbf{T}_{k+1},\ \forall\mathbf{w}_k\in\mathbf{W}_k\} \tag{5}$$

$$\mathbf{T}_k=\{\mathbf{z}\in R^n\colon\ \mathbf{f}_k(\mathbf{z}, \mathbf{u}_k)\in\mathbf{E}_{k+1}, \text{ for some } \mathbf{u}_k\in\mathbf{U}_k\} \tag{6}$$

$$\mathbf{T}_N=\mathbf{X}_N. \tag{7}$$

We then have by repeated application of statements (2) or (3) above.

Proposition 1. The target set $\mathbf{X}_N$ is reachable at time N from all points of a given set $\mathbf{X}_k$ of states $\mathbf{x}_k$ at time k if and only if $\mathbf{X}_k\subset\mathbf{T}_k$ where $\mathbf{T}_k$ is defined recursively by equations (5–7). In particular, the target set $\mathbf{X}_N$ is reachable from $\mathbf{X}_o$ at time 0 if and only if $\mathbf{X}_o\subset\mathbf{T}_o$.

It is easily seen that, as long as the target set $\mathbf{X}_N$ is reachable at time N from at least one state at time 0, the recursive relations (5–7) define two "tubes" in $R^n\times J_N$ where $J_N=\{0, 1, 2, \ldots, N\}$ is the (ordered) set of integers from 0 to N. These tubes are *the* (*updated*) *target tube* $\mathbf{T}$ defined by

$$\mathbf{T}=\{(\mathbf{z}_K, k)\in R^n\times J_N\colon\ \mathbf{z}_k\in\mathbf{T}_k,\ k=0, 1, 2, \ldots, N\} \tag{8}$$

and the *effective target tube* $\mathbf{E}$ defined by

$$\mathbf{E}=\{(\mathbf{z}_k, k)\in R^n\times J_N\colon\ \mathbf{z}_k\in\mathbf{E}_k,\ k=1, 2, \ldots, N\}. \tag{9}$$

These two tubes can be viewed as the sequence $\{\mathbf{T}_o, \mathbf{T}_1, \ldots, \mathbf{T}_{N-1}, \mathbf{T}_N\}$ of updated target sets and the sequence $\{\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_N\}$ of effective target sets. Recalling that a necessary and sufficient for $\mathbf{X}_N$ to be reachable at time N from the state $\mathbf{x}_k$, or singleton set $\{\mathbf{x}_k\}$, at time k is that $\mathbf{x}_k\in\mathbf{T}_k$, we see that once the state $\mathbf{x}_k$ at time k is inside the target tube $\mathbf{T}$ the controller can force the subsequent states $\mathbf{x}_{k+1}, \mathbf{x}_{k+2}, \ldots, \mathbf{x}_{N-1}, \mathbf{x}_N$ to stay inside the target tube, regardless of the subsequent disturbances. Thus, in particular, the final state $\mathbf{x}_N$ will be inside the target set $\mathbf{X}_N$. The controller accomplishes this by driving $\mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k)$ inside $\mathbf{E}_{k+1}$ so that $\mathbf{x}_{k+1}=\mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k)+\mathbf{g}_k(\mathbf{w}_k)$ will lie in $\mathbf{T}_{k+1}$ for all possible disturbances $\mathbf{w}_k\in\mathbf{W}_k$. This is illustrated in Fig. 1. Conversely, if $\mathbf{x}_k\notin\mathbf{T}_k$, then Nature can force the subsequent states $\mathbf{x}_{k+1}, \ldots, \mathbf{x}_{N-1}, \mathbf{x}_N$ to remain outside the target tube regardless of the permissible control action taken by the controller, and thus, in particular, the final state $\mathbf{x}_N$ can be forced to lie outside the target set $\mathbf{X}_N$.

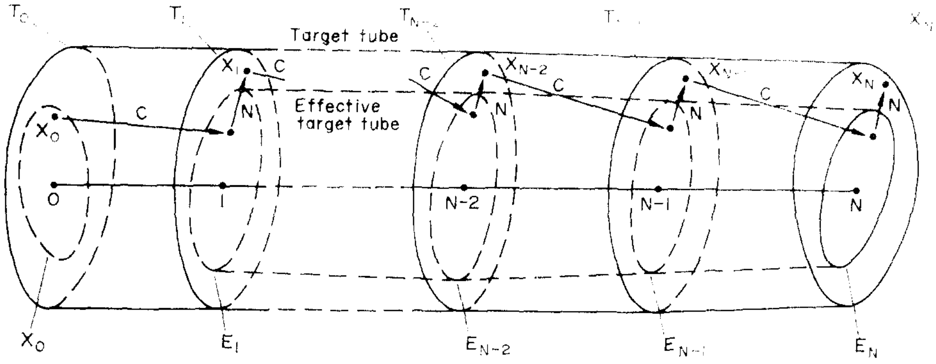

FIG. 1. Schematic presentation of the action of the controller (C) to counteract disturbances (N) for reachability of a target set.

It should be noted that both tubes are precomputable and can in principle be stored by the controller. By making additional assumptions such as linearity of the system and convexity, closure or compactness of the sets $\mathbf{U}_k$, $\mathbf{W}_k$ and $\mathbf{X}_N$, one can obtain a better characterization of the tubes, and devise algorithms for their construction. This will be discussed in Section 4.

3. REACHABILITY OF A TARGET TUBE

Consider now the extension of the problem considered in the preceding section where, instead of being concerned only that the final state of the system lie in the prescribed target set, we desire to keep the entire system trajectory within a "tube" in $R^n \times J_N$. In other words, *at each time* $K=0, 1, \ldots, N$, the system state is to be contained in a given set $\mathbf{X}_k$.

Problem 2. Consider the discrete-time dynamic system

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{g}_k(\mathbf{w}_k) \tag{1}$$

where, as in Problem 1, $\mathbf{x}_k \in R^n$, $\mathbf{u}_k \in \mathbf{U}_k \subset R^r$, $\mathbf{w}_k \in \mathbf{W}_k \subset R^p$ and $\mathbf{x}_0 \in \mathbf{X}_0$. Given a prescribed target "tube" $\{(\mathbf{X}_k, k)\colon k=1, 2, \ldots, N\} \subset R^r \times J_N$, find (if it exists) a closed-loop control law $\mathbf{u}(\cdot, \cdot)$ mapping the pairs $(\mathbf{x}_k, k)$ into $\mathbf{U}_k$, $k=0, 1, 2, \ldots, N-1$, with the property that at each time $k=1, 2, \ldots, N$, the state $\mathbf{x}_k$ of the closed-loop system

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, u(\mathbf{x}_k, k)) + \mathbf{g}_k(\mathbf{w}_k) \tag{2}$$

is contained in $\mathbf{X}_k$ for all possible disturbance sequences $\mathbf{w}_k \in \mathbf{W}_k$, $k=0, 1, 2, \ldots, N-1$, and all possible initial states $\mathbf{x}_o \in \mathbf{X}_o$.

Definition 2. The target tube

$$\{(\mathbf{X}_k, \mathbf{k}), k=1, 2, \ldots, N\} \subset R^n \times J_N$$

is reachable from the initial state set $\mathbf{X}_0$ at time 0 if there exists at least one solution to Problem 2.

We remark that by taking all of the sets $\mathbf{X}_k$ but $\mathbf{X}_N$ to be the entire space R^n, Problem 2 reduces directly to Problem 1. The results in this section are simply generalizations of those given in the preceding section in the sense that we now take into account the requirement that $\mathbf{x}_k$ be in $\mathbf{X}_k$ for all k.

It was shown in the preceding section that a necessary and sufficient condition for $\mathbf{x}_N$ to be contained in $\mathbf{X}_N$ for all $\mathbf{w}_{N-1} \in \mathbf{W}_{N-1}$ and some $\mathbf{u}_{N-1} \in \mathbf{U}_{N-1}$ is that $\mathbf{x}_{N-1}$ lie in the updated target set $\mathbf{T}_{N-1}$ defined by equations (3) and (4). Thus in order for $\mathbf{x}_N$ to be contained in $\mathbf{X}_N$ *and* for $\mathbf{x}_{N-1}$ to be contained in $\mathbf{X}_{N-1}$, it is clearly both necessary and sufficient that $\mathbf{x}_{N-1}$ be contained in both $\mathbf{T}_{N-1}$ and $\mathbf{X}_{N-1}$, i.e. that $\mathbf{x}_{N-1}$ be an element of the *modified target set* $\mathbf{X}^*_{N-1}$ at time $N-1$ defined by

$$\mathbf{X}^*_{N-1} = \mathbf{T}_{N-1} \cap \mathbf{X}_{N-1} \tag{10}$$

where $\cap$ denotes set intersection. We therefore have that the tube $\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{N-1}, \mathbf{X}_N\}$ is reachable from $\mathbf{X}_0$ at time 0 if and only if the tube $\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{N-2}, \mathbf{X}^*_{N-1}\}$ is reachable from $\mathbf{X}_0$ at time 0. In other words, the reachability of a tube of length N has been reduced to the reachability of a tube of length $N-1$. Repeated application of this procedure leads to a complete solution of the target tube reachability problem. To this end we define recursively the effective target set $\mathbf{E}^*_{k+1}$ at time $k+1$, the updated target set $\mathbf{T}^*_k$ at time k and the modified target set $\mathbf{X}^*_k$ at time k as follows, c.f. equations (5–7) and (10).

$$\mathbf{E}^*_{k+1} = \{\mathbf{z} \in R^n\colon [\mathbf{z} + \mathbf{g}_k(\mathbf{w}_k)] \in \mathbf{X}^*_{k+1}, \forall \mathbf{w}_k \in \mathbf{W}_k\} \tag{11}$$

$$\mathbf{T}^*_k = \{\mathbf{z} \in R^n\colon \mathbf{f}_k(\mathbf{z}, \mathbf{u}_k) \in \mathbf{E}^*_{k+1} \text{ for some } \mathbf{u}_k \in \mathbf{U}_k\} \tag{12}$$

$$\mathbf{X}^*_k = \mathbf{T}^*_k \cap \mathbf{X}_k \tag{13}$$

$$\mathbf{X}^*_N = \mathbf{X}_N. \tag{14}$$

We then have by repeated application of these definitions, c.f. Proposition 1:

Proposition 2. The target tube

$$\{(\mathbf{X}_j, j);\ j=k+1, \ldots, N\}$$

is reachable from state $\mathbf{x}_k$ at time k if and only if $\mathbf{x}_k \in \mathbf{T}_k^*$. In particular, the target tube $\{(\mathbf{X}_j, j)$: $j=1, 2, \ldots, N\}$ is reachable from $\mathbf{X}_0$ at time 0 if and only if $\mathbf{X}_0 \subset \mathbf{T}_0^*$.

In a manner analogous to the introduction of the updated target tube and the effective target tube in equations (8) and (9) of the preceding section, we can view the recursive equations (11–14) as defining two tubes in $R^n \times J_N$. These are the *modified target tube* **M** and the effective target tube **E*** defined via equations (11–14) by

$$\mathbf{M}=\{(\mathbf{X}_k^*, k);\ k=1, 2, \ldots, N\} \qquad (15)$$

$$\mathbf{E}^*=\{(\mathbf{E}_k^*, k);\ k=1, 2, \ldots, N\} \qquad (16)$$

which, alternatively, may be viewed as the sequences $\{\mathbf{X}_1^*, \mathbf{X}_2^*, \ldots, \mathbf{X}_{N-1}^*, \mathbf{X}_N\}$ and $\{\mathbf{E}_1^*, \mathbf{E}_2^*, \ldots, \mathbf{E}_N^*\}$ of modified and effective target sets. Once the system state lies inside the modified target tube **M** the controller can force the remaining state trajectory to lie in the desired target tube

$$\{(\mathbf{X}_k, k);\ k=1, 2, \ldots, N\}$$

regardless of disturbances. The controller accomplishes this by choosing the control at each time in such a way that $\mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k)$ lies inside $\mathbf{E}^*_{k+1}$, so that $\mathbf{x}_{k+1} \in \mathbf{X}_{k+1}^*$ for all $\mathbf{w}_k \in \mathbf{W}_k$. This is illustrated in Fig. 2. Conversely, if the initial state $\mathbf{x}_0$ does not lie in $\mathbf{T}_0^*$, then nature can select the disturbances in such a way that at least part of the trajectory lies outside the desired target tube

$$\{(\mathbf{X}_k, k):\ k=1, 2, \ldots, N\}.$$

It should be noted that, as is to be expected with a dynamic programming type of procedure, the effective and modified target tubes **E*** and **M** must be precalculated "backwards in time" and stored. Exact and approximate procedures for doing this efficiently are discussed in subsequent sections.

4. SOME RESULTS FOR LINEAR SYSTEMS

We now specialize the results of the past two sections to the case of a linear system

$$\mathbf{x}_{k+1} = \mathbf{A}_k\mathbf{x}_k + \mathbf{B}_k\mathbf{u}_k + \mathbf{G}_k\mathbf{w}_k. \qquad (17)$$

For such systems the updated target sets $\mathbf{T}^*_k$ can be defined as the inverse image under $\mathbf{A}_k$ of the set $\{\mathbf{E}^*_{k+1} + (-\mathbf{B}_k\mathbf{U}_k)\}$, i.e.

$$\mathbf{T}_k^* = \mathbf{A}_k^{-1}\{\mathbf{E}_{k+1}^* + (-\mathbf{B}_k\mathbf{U}_k)\} \qquad (18)$$

where + denotes the vector sum of the indicated sets. Note that equation (18) involves set operations only, and $\mathbf{T}_k^*$ is defined even if the matrix $\mathbf{A}_k$ is not invertible.

If $\mathbf{X}_N$ is a convex set it is easy to prove that the effective target set $\mathbf{E}_N^*$ defined by

$$\mathbf{E}_N^* = \{\mathbf{z}:\ (\mathbf{z} + \mathbf{G}_{N-1}\mathbf{w}_{N-1}) \in \mathbf{X}_N,\ \forall \mathbf{w}_{N-1} \in \mathbf{W}_{N-1}\}$$

is also convex. It is not necessary that the set $\mathbf{W}_{N-1}$ be convex. In fact, the set $\mathbf{E}_N^*$ remains unchanged if $\mathbf{W}_{N-1}$ is replaced by its convex hull. If $\mathbf{U}_{N-1}$ and $\mathbf{X}_{N-1}$ are also convex then the updated and modified target sets $\mathbf{T}_{N-1}^*$, and $\mathbf{X}_{N-1}^*$ are convex since the operations in equation (18) and set intersection preserve convexity. It is also clear that if all given sets are compact. the sets $\mathbf{E}^*_{k+1}$, $\mathbf{T}^*_k$, and $\mathbf{X}_k$, $k=0, 1, 2, \ldots, N-1$, are closed If, in,

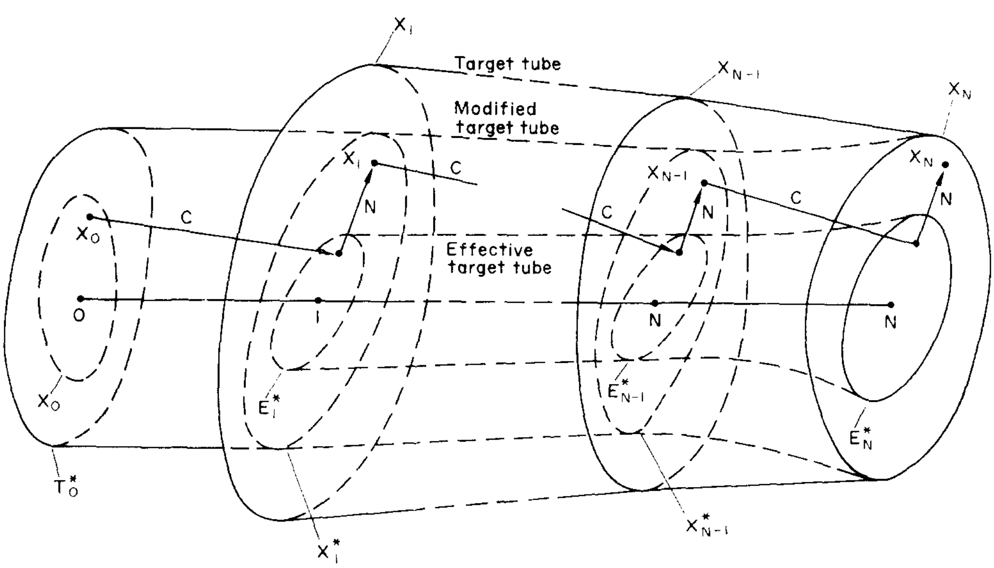

FIG. 2. Schematic presentation of the action of the controller (C) to counteract disturbances (N) for reachability of a target tube.

addition, $\mathbf{A}_k$ is invertible for all $k=0, 1, \ldots, N-1$ and all given sets are compact, the sets $\mathbf{E}^*_{k+1}$, T^*_k, $\mathbf{X}^*_k$ are also compact. We summarize the above in the following proposition:

Proposition 3. If, for the linear system (17), the sets $\mathbf{X}_{k+1}$, $\mathbf{U}_k$ are convex for all $k=0, 1, \ldots, N-1$, the sets $\mathbf{E}^*_{k+1}$, $\mathbf{T}^*_k$, and $\mathbf{X}^*_k$ defined by equations (11) through (14) are also convex for all k. If, for all $k=0, 1, 2, \ldots, N-1$, the sets $\mathbf{X}_{k+1}$, $\mathbf{W}_k$ and $\mathbf{U}_k$ are compact the sets $\mathbf{E}^*_{k+1}$, $\mathbf{T}_k$ and $\mathbf{X}^*_k$ are closed; if, in addition, the sets $\mathbf{X}_{k+1}$, $\mathbf{W}_k$ and $\mathbf{U}_k$ are compact and $\mathbf{A}_k$ is invertible for all $k=0, 1, 2, \ldots, N-1$, the sets $\mathbf{E}^*_{k+1}$, $\mathbf{T}^*_k$ and $\mathbf{X}^*_k$ are also compact for all k.

For practical applications it is important that the sets $\mathbf{E}^*_{k+1}$, $\mathbf{T}^*_k$, and $\mathbf{X}^*_k$ can be characterized by a finite set of numbers. This is possible when $\mathbf{X}_k$ and $\mathbf{U}_k$ are convex polyhedra. The sets $\mathbf{E}^*_{k+1}$, $\mathbf{T}^*_k$ and $\mathbf{X}^*_k$ are in this case polyhedra and thus can be characterized by a finite number of bounding hyperplanes. Given the state $\mathbf{x}_k$ at time k, the set of all controls $\mathbf{u}_k$ with the property that

$$(\mathbf{A}_k\mathbf{x}_k+\mathbf{B}_k\mathbf{u}_k)\in\mathbf{E}_{k+1}$$

may, under these circumstances, be determined on-line as the intersection of two polyhedra. Any control in this intersection is then adequate for reachability. The relevant algorithm is presented in Appendix 1.

If, however, the given sets are not polyhedra, the exact calculation of the modified target tube and the effective target tube given by equations (15) and (16) becomes extremely difficult if not infeasible. One can, however, conceive of constructing approximations to these tubes that are characterized by a finite set of numbers. One such possibility is to approximate the sets $\mathbf{X}^*_k$, $\mathbf{E}^*_k$ and $\mathbf{T}^*_k$ for each $K=0, 1, 2, \ldots, N$, by ellipsoids $\bar{\mathbf{X}}_k\subset X^*_k$, $\hat{\mathbf{E}}_k\subset\mathbf{E}^*_k$ and $\bar{T}_k\subset T^*_k$, since an ellipsoid is completely characterized by its center and a weighting matrix. In this way, the modified target tube $\mathbf{M}$, for example, is approximated by an internal tube $\mathbf{M}=\{(\bar{\mathbf{X}}_k, k)\colon\ k=1, \ldots, N\}$ whose cross-sections are ellipsoids. Then, in order for the original target tube $\{(\mathbf{X}_k, k)\colon\ k=1, 2, \ldots, N\}$ to be reachable from the set $\mathbf{X}_0$ at time 0, it is sufficient (but not necessary) that $\mathbf{X}_0\subset\bar{T}_0$. This approximation approach is the basis for the ellipsoidal approximation algorithm given in Appendix 2, where results on the optimal control of linear systems with quadratic cost criteria are used not only to derive ellipsoidal tubes but also to derive control laws that are *linear*.

5. RELATION BETWEEN CONTROL AND ESTIMATION ALGORITHMS

It is well-known that there exists a duality between certain stochastic estimation problems [10] and a class of optimal control problems involving a quadratic cost functional [9]. Amongst other effects, this duality is reflected in the fact that in both cases the solution involves Riccati equations. In the estimation case the solution of the Riccati equation is propagated forward in time, while in the control case the solution is propagated backwards in time. The state estimation problem for linear discrete systems where all the uncertain quantities are described by their membership in given sets has been considered by several authors [1–5]. The objective is to estimate the set of all feasible states compatible with the measurements received. Consider the case of the linear discrete system:

$$\mathbf{x}_{k+1}=\mathbf{F}_k\mathbf{x}_k+\mathbf{G}_k\mathbf{w}_k \tag{19}$$

with measurements of the form

$$\mathbf{z}_k=\mathbf{H}_k\mathbf{x}_k+\mathbf{v}_k \tag{20}$$

where $\mathbf{x}_k\in R^n$ with the initial condition $\mathbf{x}_0$ contained in a given set $\mathbf{X}_o\subset R^n$, and the input disturbance $\mathbf{w}_k$ and the measurement disturbance $\mathbf{v}_k$ are, at each time $k=1, 2, \ldots N$, contained in known sets $\mathbf{W}_k\subset R^p$ and $\mathbf{V}_k\subset R^m$. Then it can be shown [1], [2] that the set $\mathbf{S}_{k|k}$ of possible states $\mathbf{x}_k$ consistent with a given set of measurements $\mathbf{z}_1, \ldots, \mathbf{z}_k$ is given recursively by the following equations

$$\mathbf{S}_{k|k}=\mathbf{S}_{k|k-1}\cap\{\mathbf{x}_k\colon\mathbf{z}_k-\mathbf{H}_k\mathbf{x}_k\in\mathbf{V}_k\} \tag{21}$$

$$\mathbf{S}_{k|k-1}=\mathbf{F}_{k-1}\mathbf{S}_{k-1|k-1}+\mathbf{G}_{k-1}\mathbf{W}_{k-1} \tag{22}$$

$$\mathbf{S}_{0|0}=\mathbf{X}_o. \tag{23}$$

One would like to identify a "duality" relation between an estimation problem of this form and a control problem. Such a relation exists and, as we now show, the corresponding control problem is the special case of the target tube reachability problem considered in section 3 where there is no input noise.

Consider the special case of Problem 2 in which the system is linear and there is no disturbance in the dynamics, so that the system equation is

$$\mathbf{x}_{k+1}=\mathbf{A}_k\mathbf{x}_k+\mathbf{B}_k\mathbf{u}_k. \tag{24}$$

In this case we wish to keep the system output

$$\mathbf{y}_k=\mathbf{C}_k\mathbf{x}_k \tag{25}$$

in a prescribed tube $\{(\mathbf{Y}_k, k);\ k=1, 2, \ldots, N\}$ in $R^n\times J_N$ by the appropriate choice of control law

$\mathbf{u}(\cdot, \cdot)$ mapping the pairs $(\mathbf{x}_k, k)$ into $\mathbf{U}_k$. Using the results of Sections 3 and 4, the corresponding modified target tube $\mathbf{X}_k^*$ is generated by the algorithm:

$$\mathbf{X}_k^* = \mathbf{T}_k^* \cap \{\mathbf{x}_k : \mathbf{C}_k \mathbf{x}_k \in \mathbf{Y}_k\} \tag{26}$$

$$\mathbf{T}_k^* = \mathbf{A}_k^{-1} \mathbf{X}_{k+1}^* + (-\mathbf{A}_k^{-1} \mathbf{B}_k) \mathbf{U}_k \tag{27}$$

where

$$\mathbf{X}_N^* = \{\mathbf{x}_N : \mathbf{C}_N \mathbf{x}_N \in \mathbf{Y}_N\}. \tag{28}$$

It can be seen that the algorithm (26–28) for the control problem and the algorithm (21–23) for the estimation problem have certain similarities. They both have at each step a set intersection involving the output, and a vector sum operation involving the input. The solution in the case of the estimation algorithm propagates forward in time whereas in the case of the control algorithm it propagates backwards. In fact, if between the systems (19), (20) and (24), (25) we make the identifications

$$\mathbf{F}_{k-1} = \mathbf{A}_{N-k}^{-1}; \quad \mathbf{G}_{k-1} = -\mathbf{A}_{N-k}^{-1} \mathbf{B}_{N-k};$$
$$\mathbf{H}_k = -\mathbf{C}_{N-k} \tag{29}$$

and between the corresponding sets involved we make the identifications

$$\mathbf{X} = \mathbf{X}_N^* = \{\mathbf{z} : \mathbf{C}_N \mathbf{z} \in \mathbf{Y}_N\}; \quad \mathbf{V}_k = \mathbf{Y}_{N-k};$$
$$\mathbf{W}_{k-1} = \mathbf{U}_{N-k} \tag{30}$$

then, for the special case where the measurements are zero (i.e. $\mathbf{z}_1 = \ldots = \mathbf{z}_k = 0$), we have by comparing (21–23) with (26–28),

$$\mathbf{S}_{k|k} = \mathbf{X}^*_{N-k} \quad k = 0, \ldots, N-1. \tag{31}$$

Thus one can solve the control problem by solving the corresponding estimation problem. In either case, of course, one would like to be able to describe the sets $\mathbf{S}_{k|k}$ or $\mathbf{X}_k^*$ by finite sets of numbers, which will be true for ellipsoids, for example. However, even if all the given sets are ellipsoids, the sets $\mathbf{S}_{k|k}$ are not ellipsoids. On the other hand, lower and upper ellipsoidal bounds $\mathbf{S}_{l,k}$, $\mathbf{S}_{u,k}(\mathbf{S}_{l,k} \subset \mathbf{S}_{k|k} \subset \mathbf{S}_{u,k})$ can be calculated for them [5]. For the control problem, the lower bound is of interest as it provides suboptimal modified target sets, and it can form the basis for an ellipsoidal algorithm (with an appropriate modification for the input noise case) for construction of a suboptimal solution. However, this same algorithm can be studied best by relating the tube problem to the linear quadratic optimal control problem as is done in Appendix 2, where, in addition, linear control laws are derived.

The identifications in equations (29) and (30) are not the ones usually associated with the duality between the stochastic filtering problem and the linear system-quadratic cost optimal control problem. We remark, however, that the "usual" identifications are not the only ones for which the Riccati equations, or their discrete-time counterparts, associated with the filtering problem and the regulator problem can be put in one–one correspondence; an alternative set of identifications is given by equation (29). In fact, let $\mathbf{P}_k$ be the solution at time k of the discrete Riccati equation corresponding to the optimal control problem involving system (24) and the quadratic cost functional

$$J[\mathbf{u}] = \mathbf{x}_N' \mathbf{\Psi}^{-1} \mathbf{x}_N + \sum_{i=0}^{N-1} [\mathbf{x}_i' \mathbf{C}_i' \mathbf{R}_i^{-1} \mathbf{C}_i \mathbf{x}_i + \mathbf{u}_i' \mathbf{Q}_i^{-1} \mathbf{u}_i]$$

and, with $\mathbf{\Psi}$, $\mathbf{R}_i$, $\mathbf{Q}_i$ positive definite matrices, let $\Sigma(k|k)$ be the solution at time k of the Riccati equation corresponding to the stochastic filtering problem involving the system (19), (20) with x_o, w_{k-1}, v_k $(k=1, \ldots N)$ being independent Gaussian random vectors with zero mean and covariances equal to $\mathbf{\Psi}$, $\mathbf{Q}_{N-k-1}$, $\mathbf{R}_{N-k}$ respectively. Then, by writing the corresponding equations, it can be easily seen that under the identifications (29) we have

$$\Sigma^{-1}(k|k) = \mathbf{P}_{N-k} \qquad (k = 0, \ldots, N).$$

6. REACHABILITY WITH IMPERFECT STATE INFORMATION

In this section we extend Problems 1 and 2 to the case where, instead of having perfect knowledge of the system state, the controller has access only to noise-corrupted measurements of the system output. The objective is again either to drive the state $\mathbf{x}_N$ of the system inside a target set at the final time or to keep the entire state trajectory inside a target tube. We restrict attention to linear systems and assume that all given sets are convex. Within these assumptions, we derive sufficient conditions for reachability. The complete solution of the problem is given in principle by Dynamic Programming [1]; however, it appears to involve all the complexities of a dual control problem [8]. We depart here from a strict Dynamic Programming formulation. For this reason the conditions we derive are only sufficient and the results are weaker than those of the perfect information case.

Consider again the linear system

$$\mathbf{x}_{k+1} = \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u}_k + \mathbf{G}_k \mathbf{w}_k \tag{32}$$

where now the controller has available only measurements of the form

$$\mathbf{z}_k = \mathbf{C}_k \mathbf{x}_k + \mathbf{v}_k \quad (33)$$

and the observation noise vectors $\mathbf{v}_k$ are known to belong to given bounded sets $\mathbf{V}_k$. Assume that the sets $\mathbf{W}_k$, $\mathbf{V}_k$ and $\mathbf{X}_o$ are the ellipsoids

$$\begin{aligned} \mathbf{W}_k &= \{\mathbf{w}_k: \ \mathbf{w}_k'\mathbf{Q}_k^{-1}\mathbf{w}_k \le 1\} \\ \mathbf{V}_k &= \{\mathbf{v}_k: \ \mathbf{v}_k'\mathbf{R}_k^{-1}\mathbf{v}_k \le 1\} \\ \mathbf{X}_o &= \{\mathbf{x}_o: \ (\mathbf{x}_o - \boldsymbol{\mu}_o)'\psi^{-1}(\mathbf{x}_o - \boldsymbol{\mu}_o) \le 1\} \end{aligned} \quad (34)$$

where $\mathbf{Q}_k$, $\mathbf{R}_k$ and $\mathbf{\Psi}$ are positive definite matrices and $\boldsymbol{\mu}_o$ is a known n-vector.

Given at time k the measurements $\mathbf{z}_1, \ldots, \mathbf{z}_k$ and the prior controls $\mathbf{u}_o, \ldots, \mathbf{u}_{k-1}$, the controller can in principle estimate the set of possible states $\mathbf{x}_k$ compatible with the measurements. Unfortunately, however, this set is not easily characterized or computed in practice; on the other hand, an ellipsoidal bound to it can readily be calculated. We give the relevant algorithm below in Proposition 4, the proof of which can be found in Ref. [5]. This algorithm is formally identical to a stochastic Kalman estimator in modified form and bears close relation to an algorithm due to Schweppe [2, 3]. However, it has important advantages over the latter as the resulting estimator is linear, has precomputable gains and as time approaches infinity, it converges to a time-invariant filter. Since the effect of a known control can be superimposed, we give the algorithm assuming $\mathbf{u}_k \equiv 0$.

Proposition 4. An ellipsoidal estimate set $\mathbf{\Omega}(k)$ which contains the set of possible states $\mathbf{x}_k$ of the system (32) compatible with observed measurements $\mathbf{z}_1, \ldots, \mathbf{z}_k$ is given by:

$$\mathbf{\Omega}(k) = \{\mathbf{x}_k : (\mathbf{x}_k - \hat{\mathbf{x}}_k)'\mathbf{\Sigma}^{-1}(k|k)(\mathbf{x}_k - \hat{\mathbf{x}}_k) \le 1 - \delta^2(k)\} \quad (35)$$

where the center $\hat{\mathbf{x}}_k \in R^n$ of the ellipsoid is given recursively by

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}_k\hat{\mathbf{x}}_k + \rho_{k+1}\mathbf{\Sigma}(k+1|k+1) \mathbf{C}_{k+1}'\mathbf{R}_{k+1}^{-1}(\mathbf{z}_{k+1} - \mathbf{C}_{k+1}A_k\hat{\mathbf{x}}_k) \quad (36)$$

the $n \times n$ weighting matrix $\mathbf{\Sigma}(k|k)$ of the ellipsoid is given recursively by

$$\mathbf{\Sigma}(k|k) = [(1-\rho_k)\mathbf{\Sigma}^{-1}(k|k-1) + \rho_k\mathbf{C}_k'\mathbf{R}_k^{-1}\mathbf{C}_k]^{-1} \quad (37)$$

$$\mathbf{\Sigma}(k|k-1) = (1-\beta_{k-1})^{-1}\mathbf{A}_{k-1}\mathbf{\Sigma}(k-1|k-1)\mathbf{A}'_{k-1} + \beta_{k-1}^{-1}\mathbf{G}_{k-1}\mathbf{Q}_{k-1}\mathbf{G}_{k-1}' \quad (38)$$

and the scalar term $\delta^2(k)$ is given by

$$\delta^2(k) = (1-\beta_{k-1})(1-\rho_k)\delta^2(k-1) + \delta^2(z_k) \quad (39)$$

where

$$\delta^2(z_k) = (\mathbf{z}_k - \mathbf{C}_k\mathbf{A}_{k-1}\hat{\mathbf{x}}_{k-1})'[(1-\rho_k)^{-1}\mathbf{C}_k\mathbf{\Sigma}(k|k-1)\mathbf{C}_k' + \rho_k^{-1}\mathbf{R}_k]^{-1}(\mathbf{z}_k - \mathbf{C}_k\mathbf{A}_{k-1}\hat{\mathbf{x}}_{k-1}). \quad (40)$$

The initial conditions are:

$$\hat{\mathbf{x}}_o = \boldsymbol{\mu}_o, \qquad \mathbf{\Sigma}(0|0) = \mathbf{\Psi}, \qquad \delta^2(0) = 0 \quad (41)$$

while β_{k-1} and ρ_k, $k = 1, \ldots, N$, are arbitrary parameters with $0 < \beta_{k-1} < 1$ and $0 < \rho_k < 1$.

It can be seen that the weighting matrix $\mathbf{\Sigma}(k|k)$ of the estimate ellipsoid is precomputable and that the lengths of its axes are proportional to the square root of the term $[1-\delta^2(k)]$ which depends via equations (39) and (40), on the particular measurements received. Since $\delta^2(k) \ge 0$, one can always precompute (except for the center) the largest possible estimate set

$$\mathbf{\Omega}_k = \{\mathbf{x}_k: \ (\mathbf{x}_k - \hat{\mathbf{x}}_k)'\mathbf{\Sigma}^{-1}(k|k)(\mathbf{x}_k - \hat{\mathbf{x}}_k) \le 1\}. \quad (42)$$

Returning now to the reachability problem, we assume that the controller has available an estimator that gives at each time k the ellipsoidal estimate set (42) as described by its center $\hat{\mathbf{x}}_k$, which is computed on-line via (36), and its weighting matrix $\mathbf{\Sigma}(k|k)$, which may be either precomputed and stored or computed on-line. Furthermore, we restrict attention to control laws $\mathbf{u}(\cdot, \cdot)$ that map the pairs $(\hat{\mathbf{x}}_k, k)$ to $\mathbf{U}_k$. We have thus assumed that the control process may be separated from the estimation process. We now proceed to derive sufficient conditions for reachability of a target tube $\{(\mathbf{X}_k, k): \ k = 1, 2, \ldots, N\} \subset R^n \times J_N$. The approach we will follow is to reduce the target tube reachability problem with imperfect information to a target tube problem with *perfect* information, a problem that can be solved using the results of Sections 2, 3 and 4. This reduction is achieved by shifting emphasis from the reachability of a target tube by the system *state* $\mathbf{x}_k$ to the reachability of a different target tube by the *state estimate* $\hat{\mathbf{x}}_k$, a process that is possible because, once $\hat{\mathbf{x}}_k$ is known, the system state is guaranteed to lie within the set $\mathbf{\Omega}_k$ defined by equation (42). In fact, suppose we define the ellipsoid $\mathbf{S}_k$, $k = 1, 2, \ldots, N$, to be the estimate ellipsoid (42) translated to have its center at the origin, i.e.

$$\mathbf{S}_k = \{\mathbf{z}: \ \mathbf{z}'\mathbf{\Sigma}^{-1}(k|k)\mathbf{z} \le 1\}. \quad (43)$$

Notice that $\mathbf{S}_k$ is precomputable. Then it is clear that if the state estimate $\hat{\mathbf{x}}_k$ is known then the set of possible system states is contained in the ellipsoid $\hat{\mathbf{x}}_k + \mathbf{S}_k$, which is merely the estimate set (42). Conversely, in order for the system state $\mathbf{x}_k$ to lie for all k in the given target tube

$$\{(\mathbf{X}_k, k):\ k=1, 2, \ldots, N\},$$

it is sufficient that the state estimate $\hat{\mathbf{x}}_k$ lies for all k in the tube $\{(\hat{\mathbf{X}}_k, k):\ k=1, \ldots, N\}$ where the sets $\hat{\mathbf{X}}_k$ are defined as

$$\hat{\mathbf{X}}_k = \{\mathbf{z}:\ (\hat{\mathbf{x}}_k + \mathbf{z}) \in \mathbf{X}_k,\ \forall \mathbf{z} \in \mathbf{S}_k\}. \tag{44}$$

Now, substitution of equations (32) and (33) into (36) shows that the estimate $\hat{\mathbf{x}}_k$ is generated recursively by

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}_k \hat{\mathbf{x}}_k + \mathbf{B}_k \mathbf{u}_k + \mathbf{d}_k \tag{45}$$

where the lumped disturbance $\mathbf{d}_k$ is given by

$$\mathbf{d}_k = \mathbf{L}_{k+1}\mathbf{C}_{k+1}\mathbf{A}_k(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \mathbf{L}_{k+1}\mathbf{C}_{k+1}\mathbf{G}_k\mathbf{w}_k + \mathbf{L}_{k+1}\mathbf{v}_{k+1} \tag{46}$$

and the, precomputable, gain matrix $\mathbf{L}_k$ is given by

$$\mathbf{L}_k = \rho_k \mathbf{\Sigma}(k|k)\mathbf{C}_k'\mathbf{R}_k^{-1}. \tag{47}$$

Furthermore, it follows immediately from equation (46) that $\mathbf{d}_k$ belongs to the known set

$$\mathbf{D}_k = \mathbf{L}_{k+1}\mathbf{C}_{k+1}\mathbf{A}_k\mathbf{S}_k + \mathbf{L}_{k+1}\mathbf{C}_{k+1}\mathbf{G}_k\mathbf{W}_k + \mathbf{L}_{k+1}\mathbf{V}_{k+1} \tag{48}$$

where $\mathbf{S}_k$ is defined by equation (43) and the ellipsoids $\mathbf{W}_k$ and $\mathbf{V}_{k+1}$ are defined in equation (34).

Thus, *a sufficient condition for the reachability of the target tube* $\{(\mathbf{X}_k, k):k=1, \ldots N\}$ *by the system state* $\mathbf{x}_k$ *in the presence of imperfect information is that the target tube* $\{(\hat{\mathbf{X}}_k, k):k=1, \ldots, N\}$ defined by equation (44) be reachable by the state $\hat{\mathbf{x}}_k$ of the estimator (45). Since the estimate $\hat{\mathbf{x}}_k$ is generated by the controller and known to him at each time k, this problem is simply the target tube reachability problem with perfect information that was examined in sections 3 and 4.

We summarize the above development by stating the following problem and its solution:

Problem 3. Consider the discrete system (45) with the initial condition $\hat{\mathbf{x}}_o = \boldsymbol{\mu}_o$ and the target tube $\{(\hat{\mathbf{X}}_k, k):k=1, \ldots, N\}$ given by equation (44). Find, if it exists, a control law $\mathbf{u}(\cdot, \cdot)$ mapping the pairs $(\hat{\mathbf{x}}_k, k)$ into $\mathbf{U}_k$, $k=0, \ldots, N-1$, such that the state $\hat{\mathbf{x}}_k$ of system (45) lies for all k in the target tube $\{(\hat{\mathbf{X}}_k, k):k=1, \ldots, N\}$ for all possible disturbances $\mathbf{d}_k \in \mathbf{D}_k$, where the set $\mathbf{D}_k$ is given for all $k=0, 1, \ldots, N-1$ by equation (48).

The solution of Problem 3 can be given using the results of section 2. Define, analogously to equations (15), (16) and (17), the effective target set $\hat{\mathbf{E}}^*_{k+1}$ at time $k+1$, and the updated target set $\hat{\mathbf{T}}^*_k$ at time k

$$\hat{\mathbf{E}}^*_{k+1} = \{\mathbf{z} \in R^n:\ (\mathbf{z} + \mathbf{d}_k) \in \hat{\mathbf{X}}^*_{k+1},\ \forall \mathbf{d}_k \in \mathbf{D}_k\} \tag{49}$$

$$\hat{\mathbf{T}}^*_k = \{\mathbf{z} \in \mathbf{R}^n:\ (\mathbf{A}_k\mathbf{z} + \mathbf{B}_x\mathbf{u}_k) \in \hat{\mathbf{E}}^*_{k+1} \text{ for some } \mathbf{u}_k \in \mathbf{U}_k\} \tag{50}$$

$$\hat{\mathbf{X}}^*_k = \hat{\mathbf{T}}^*_k \cap \hat{\mathbf{X}}_k \tag{51}$$

$$\hat{\mathbf{X}}^*_N = \hat{\mathbf{X}}_N. \tag{52}$$

Then, by Proposition 1, a necessary and sufficient condition for the existence of a solution to Problem 3 is that $\hat{\mathbf{x}}_o = \boldsymbol{\mu}_o \in \hat{T}^*_o$, where $\boldsymbol{\mu}_o$ is defined in equation (34). Since existence of a solution of Problem 3 is, as indicated earlier, sufficient for existence of a solution to the problem of reachability of the target tube $\{(\mathbf{X}_k, k):\ k=1, \ldots, N\}$ by the state $\mathbf{x}_k$ of a system (32) in the presence of the imperfect measurements (33), we have the following proposition:

Proposition 5. A sufficient condition for reachability of the target tube $\{(\mathbf{X}_k, k):\ k=1, \ldots, N\}$ by the state $\mathbf{x}_k$ of system (32) from the initial condition set $\mathbf{X}_o$ is that $\hat{\mathbf{x}}_o = \boldsymbol{\mu}_o \in \hat{\mathbf{T}}^*$ where the set $\hat{\mathbf{T}}^*_o$ is defined recursively by equations (49–52).

As in sections 3 and 4, the effective and modified target sets $\mathbf{E}^*_{k+1}$ and $\hat{\mathbf{X}}^*_k$ are precomputable via equations (49–52), and the polyhedral algorithm of Appendix 1, and the ellipsoidal algorithm of Appendix 2 are applicable for their calculation. We also remark that the problem of reachability of a target set $\mathbf{X}_N$ in the presence of the imperfect measurements (33) can be viewed as the special case of the problem of reachability of the target tube $\{(\mathbf{X}_k, k):\ k=1, \ldots, N\}$ of this section where we take all of the sets $\mathbf{X}_k$ but $\mathbf{X}_N$ to be the entire space R^n.

It should be noted that in the derivation of the sufficient condition of Proposition 5 we have made several weakening assumptions. We have assumed that the estimate sets available to the controller are the ellipsoids given by equation (42) whereas in fact the controller can in principle calculate smaller estimate sets. In addition, in equations (45–48) we have assumed that the estimation error $(\mathbf{x}_k - \hat{\mathbf{x}}_k)$ at time k, can be any vector in the estimate set $\mathbf{S}_k$ of equation (43) whereas the set of possible values of estimation error is a subset of $\mathbf{S}_k$ which depends on the previous disturbances $\mathbf{w}_{i-1}$, $\mathbf{v}_i$ $(i=1, \ldots, k)$. Thus it is to be expected that other, possibly stronger, sufficient conditions besides the one of Proposition 5 exist. It appears, however, that such conditions would require sizable on-line computations that would make the control scheme impractical or even infeasible. On the other hand, the implementation of a control scheme based on the sufficient condition of Proposition 5 presents no more difficulty than the one of the perfect information case.

7. APPLICATION TO DIFFERENTIAL GAMES

In this section we indicate how, with minor modifications, the results obtained in previous sections may be applied to the examination of a class of differential games. Consider again the linear discrete-time system

$$\mathbf{x}_{k+1} = \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u}_k + \mathbf{G}_k \mathbf{w}_k \qquad (52)$$

where in this case we identify the controller selecting the control $\mathbf{u}_k$, $k=0, 1, 2, \ldots, N-1$, as "the evader" and the controller selecting $\mathbf{w}_k$ as "the pursuer". The initial state $\mathbf{x}_o$ is assumed known to both controllers, as is the state $\mathbf{x}_k$ as it evolves in time. As before, the controllers are constrained to select control laws $\mathbf{u}_k(\cdot)$ and $\mathbf{w}_k(\cdot)$ whose values lie, respectively, in the prescribed sets $\mathbf{U}_k$ and $\mathbf{W}_k$, $k=0, 1, 2, \ldots, N-1$. Consider also a given *escape tube*

$$\mathbf{T}_E = \{(\mathbf{X}_k, k):\ k=0, 1, 2, \ldots, N\} \subset R^n \times J_N \quad (53)$$

and its complement in $R^n \times J_N$, the *capture tube*

$$\mathbf{T}_C = \{(\mathbf{X}_k, \mathbf{k}):\ k=0, 1, 2, \ldots, N\} = \overline{\mathbf{T}}_E \quad (54)$$

where the bar $^-$ denotes set complementation. The objective of the pursuer is to drive the system state into the capture tube $\mathbf{T}_C$, while the evader's objective is to keep the state outside the capture tube for all time, i.e. the evader attempts to keep the state trajectory in the escape tube.

An example where such a problem can arise is the case of two separate dynamic systems, an evader

$$\mathbf{y}_{k+1} = \mathbf{D}_k \mathbf{y}_k + \mathbf{E}_k \mathbf{u}_k$$

and a pursuer

$$\mathbf{z}_{k+1} = \mathbf{F}_k \mathbf{z}_k + \mathbf{H}_k \mathbf{v}_k$$

and capture occurs if the states $\mathbf{y}_k$ and $\mathbf{z}_k$ are sufficiently "close" for some k. For example, capture might be considered to occur if

$$\|\mathbf{C}_k(\mathbf{y}_k - \mathbf{z}_k)\| < \varepsilon \qquad \text{for any } k=0, 1, \ldots N.$$

By making the identifications

$$\mathbf{x}_k = \begin{bmatrix} \mathbf{y}_k \\ \mathbf{z}_k \end{bmatrix}, \qquad \mathbf{A}_k = \begin{bmatrix} \mathbf{D}_k & \mathbf{O} \\ \mathbf{O} & \mathbf{F}_k \end{bmatrix},$$

$$\mathbf{B}_k = \begin{bmatrix} \mathbf{E}_k \\ \mathbf{O} \end{bmatrix}, \qquad \mathbf{G}_k = \begin{bmatrix} \mathbf{O} \\ \mathbf{H}_k \end{bmatrix}$$

and

$$\overline{\mathbf{X}}_k = \{\mathbf{x}_k = \|\mathbf{M}_k \mathbf{x}_k\| < \varepsilon\}$$

where

$$\mathbf{M}_k = [\mathbf{C}_k, \ -\mathbf{C}_k]$$

the problem reduces to that stated above.

Returning to the original system (52), it is clear that, since the objective of the evader is to keep the state trajectory of the system inside the escape tube throughout the whole time interval, the problem from the evader's viewpoint is simply that of the reachability of the escape tube $\mathbf{T}_E$. This, in turn, is simply Problem 2 of section 3, where the evader and pursuer are identified, respectively, with the controller and nature. Recalling that the target tube $\{(\mathbf{X}_k, \mathbf{k});\ k=0, 1, 2, \ldots, N\}$ is reachable from state $\mathbf{x}_o$ at time 0 if and only if $\mathbf{x}_o$ is an element of the modified target set $\mathbf{X}_o^*$ defined by equations (11–14), it follows that escape is guaranteed for the evader if and only if the initial system state lies in $\mathbf{X}_o^*$. More generally, the modified target tube

$$M_E = \{(\mathbf{X}_k^*, k);\ k=0, 1, \ldots, N\} \qquad (55)$$

defined recursively by (11–14), is the set of all statetime pairs for which escape is guaranteed.

From the point of view of the pursuer, however, the problem is different, since for capture to occur it is sufficient that the trajectory enter the capture tube *only once* during the time interval. In other words, the pursuer is interested in the non-reachability of the escape tube, which occurs if the trajectory enters the capture tube at least once during the time interval, rather than reachability of the capture tube, for which it is demanded that the entire state trajectory lie in the capture tube. Furthermore, in order to *guarantee* capture, the pursuer must assume the pessimistic attitude of "playing first", in the sense of declaring his strategy to the evader. In other words, the problem of guaranteed capture is the problem of non-reachability of the escape tube when the evader chooses his strategy with knowledge of the pursuer's strategy. This is again Problem 2 of section 3 with the order of selecting controls reversed, i.e. the sequence of selections is: (1) Pursuer selects $\mathbf{w}_o$, (2) Evader selects $\mathbf{u}_o, \ldots,$ $(2N-1)$ Pursuer selects $\mathbf{w}_{N-1}$, $(2N)$ Evader selects $\mathbf{u}_{N-1}$. In the same way that we recursively defined the effective and modified target sets at each time k via equations (11–14), we can define their analogs in this case where the order of selections is reversed, viz.

$$\mathbf{E}^{*\prime}_{k+1}=\mathbf{X}^{*\prime}_{k+1}+(-\mathbf{B}_k\mathbf{U}_k) \tag{56}$$

$$\mathbf{T}^{*\prime}_k=\{\mathbf{z}\in R^n:\ \mathbf{A}_k\mathbf{z}+\mathbf{G}_k\mathbf{w}_k\in\mathbf{E}^{*\prime}_{k+1},\ \forall\mathbf{w}_k\in\mathbf{W}_k\} \tag{57}$$

$$\mathbf{X}^{*\prime}_k=\mathbf{X}_k\cap\mathbf{T}^{*\prime}_k \tag{58}$$

$$\mathbf{X}^{*\prime}_N=\mathbf{X}_N \tag{59}$$

Reachability of the escape tube $\mathbf{T}_E=\{(\mathbf{X}_k,\ k);\ k=0,\ 1,\ 2,\ \ldots,\ N\}$ from state $\mathbf{x}_o$ at time 0 with this reversed order of selections is, clearly by analogy with proposition 2, equivalent to $\mathbf{x}_o\in\mathbf{X}^{*\prime}_o$. Thus the escape tube $\mathbf{T}_E$ is non-reachable, and therefore capture is guaranteed, from state $\mathbf{x}_o$ at time t_o if and only if $\mathbf{x}_o\notin\mathbf{X}^{*\prime}_o$, i.e. $\mathbf{x}_o\in\overline{\mathbf{X}}^{*\prime}_o$ where, as before, the bar $\bar{\ }$ denotes set complementation. Furthermore, we can view equations (56–59) as defining a modified target tube

$$\mathbf{M}_C=\{(\mathbf{X}^*_k,k):\ k=0,1,\ldots,N\} \tag{60}$$

whose complement is the set of all state-time pairs for which capture is guaranteed.

Thus the two modified target tubes $\mathbf{M}_E$ and $\mathbf{M}_C$ defined by equations (59) and (60) may be viewed as dividing the trajectory space $R^n\times J_N$ into three regions, as shown schematically in Fig. 3. The modified target tube $\mathbf{M}_E$ is the region from which escape is guaranteed, the complement $\overline{\mathbf{M}}_C$ of the modified target tube $\mathbf{M}_C$ is the region from which capture is guaranteed, and the set of points that are in neither $\mathbf{M}_E$ nor $\mathbf{M}_C$ is the region from which neither capture nor escape is guaranteed.

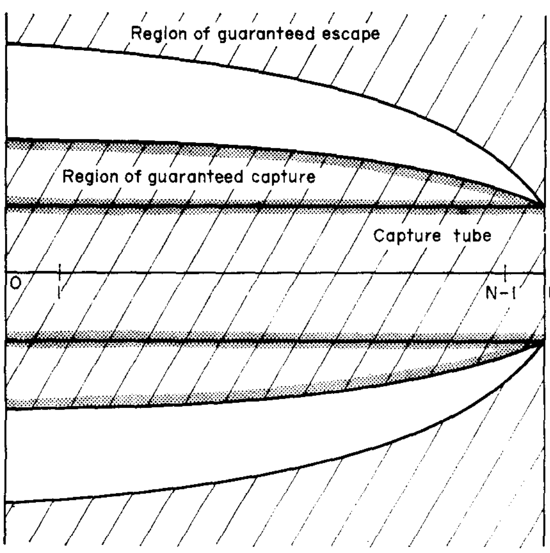

FIG. 3. Schematic presentation of the regions of guaranteed capture and guaranteed escape in a pusuit–evasion game.

It is clear that $\mathbf{M}_E$ will be a subset of $\mathbf{M}_C$ since $\mathbf{M}_C$ is the set of points in $R^n\times J_N$ from which the evader can escape capture when he "plays last" whereas $\mathbf{M}_E$ is the set of points from which the evader can escape capture when he is in the less advantageous position of having to play first, i.e. when he must declare his strategy to the pursuer. Furthermore, $\mathbf{M}_E$ will in general be a strict subset of $\mathbf{M}_C$, so that the region $\mathbf{M}_E$ of guaranteed escape and the region $\mathbf{M}_C$ of guaranteed capture are in general disjoint except at time N. This can be seen by examining the updated target sets $\mathbf{T}^*_{N-1}$ and $\mathbf{T}^{*\prime}_{N-1}$ at time $N-1$ defined by equations (11–14) and (56–59), viz.

$$\mathbf{T}^*_{N-1}=\{\mathbf{x}_{N-1}:\exists\ \mathbf{u}_{N-1}\in\mathbf{U}_{N-1}\text{ s.t. }\forall\mathbf{w}_{N-1}\in\mathbf{W}_{N-1},\ \mathbf{A}_{N-1}\mathbf{x}_{N-1}+\mathbf{B}_{N-1}\mathbf{u}_{N-1}+\mathbf{G}_{N-1}\mathbf{w}_{N-1}\in\mathbf{X}_N\} \tag{61}$$

$$\mathbf{T}^{*\prime}_{N-1}=\{\mathbf{x}_{N-1}:\forall\mathbf{w}_{N-1}\in\mathbf{W}_{N-1},\ \exists\mathbf{u}_{N-1}\in\mathbf{U}_{N-1}\text{ s.t. }\mathbf{A}_{N-1}\mathbf{x}_{N-1}+\mathbf{B}_{N-1}\mathbf{u}_{N-1}+\mathbf{G}_{N-1}\mathbf{w}_{N-1}\in\mathbf{X}_N\} \tag{62}$$

it is clear that in order for $\mathbf{T}^*_{N-1}$ to equal $\mathbf{T}^{*\prime}_{N-1}$, the order of the phrases "$\exists\mathbf{u}_{N-1}\in\mathbf{U}_{N-1}$" and "$\forall\mathbf{w}_{N-1}\in\mathbf{W}_{N-1}$" must be interchangeable, which is not in general the case.

The three regions in $R^n\times J_N$ of guaranteed capture, guaranteed escape, and neither guaranteed capture nor guaranteed escape can be interpreted profitably in terms of a sequential zero-sum game involving the system (52) and the cost functional

$$J(\mathbf{x}_k,\ k,\ \mathbf{u},\ \mathbf{v})=\begin{cases}1 & \text{if the evader escapes}\\ 0 & \text{if the evader is captured.}\end{cases} \tag{63}$$

This is simply the characteristic function of the escape tube (54) in $R^n\times J_N$. It is clear that the evader wishes to maximize J and the pursuer wishes to minimize J.

A moment's reflection shows that the region of guaranteed escape is the set of state-time pairs $(\mathbf{x}_k,\ k)$ for which

$$\max_{\mathbf{u}}\min_{\mathbf{w}} J[\mathbf{x}_k,k,\ \mathbf{u},\ \mathbf{v}]=\min_{\mathbf{w}}\max_{\mathbf{u}} J[\mathbf{x}_k,\ k,\ \mathbf{u},\ \mathbf{v}]=1$$

i.e. the set of state-time pairs for which the upper and lower value of the game are both equal to 1. Similarly, the region of guaranteed capture is the set of state-time pairs for which

$$0=\max_{\mathbf{u}}\min_{\mathbf{w}} J[\mathbf{x}_k,\ k,\ \mathbf{u},\ \mathbf{v}]=\min_{\mathbf{w}}\max_{\mathbf{u}} J[\mathbf{x}_k,\ k,\ \mathbf{u},\ \mathbf{v}].$$

The region for which neither capture nor escape are guaranteed is the set of $(\mathbf{x}_k, k)$ for which

$$0 = \max_{\mathbf{u}} \min_{\mathbf{w}} J[\mathbf{x}_k, k, \mathbf{u}, \mathbf{v}] < \min_{\mathbf{w}} \max_{\mathbf{u}} J[\mathbf{x}_k, k, \mathbf{u}, \mathbf{v}] = 1$$

i.e. for which the game has no saddle point in pure strategies. Under these conditions, one might wish to proceed in a number of ways. The usual procedure is to seek a saddle point in mixed strategies. We do not investigate this situation further in this paper.

It should be noted that for a constant system where the sets $\mathbf{U}_k$, $\mathbf{W}_k$, $\mathbf{X}_k$ are also constant one can determine the minimum time for guaranteed capture from a given initial condition $\mathbf{x}_0$. This minimum time is $(N-q)$ where q is the largest time index of sets $\overline{\mathbf{X}}_k^{*\prime}$ that contain $\mathbf{x}_0$.

We finally remark that the polyhedral algorithm of Appendix 1 is applicable for characterization of the guaranteed capture and guaranteed escape tubes when the sets $\mathbf{U}_k$, W_k, and $\mathbf{X}_k$ are polyhedra or unions of disjoint and closed polyhedra. In the particular case where $\mathbf{X}_1 = \mathbf{X}_2 = \ldots \mathbf{X}_{N-1} = \mathbf{X} = R^n$ and the problem is closely related to the target set reachability problem the computational requirements are greatly reduced.

8. CONCLUSIONS

Attention has been given to the problem of the reachability of a target set or a target tube by the state of a discrete dynamic system. Necessary and sufficient conditions for existence of a solution are given for the case where the state of the system can be measured exactly, while sufficient conditions for existence of a solution are given for the case when only disturbance-corrupted output measurements are available. Algorithms for implementation of the relevant control schemes are given for the case of a linear system; in particular, the ellipsoidal approximation algorithm given in Appendix 2 leads to linear control laws. It is also shown how the target tube reachability problem is related to a class of pursuit–evasion games.

The results reported in this paper can be extended in several ways. The problems of reachability of a target set and a target tube for a continuous time system, and particularly the problem of infinite time reachability for both discrete and continuous time systems deserve attention. For this latter problem some results have been reported in this paper in connection with the ellipsoidal algorithm of Appendix 2. However, the infinite time reachability problem is essentially different in structure from the problems considered in this paper, and it will be the subject of a forthcoming publication.

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APPENDIX 1

A polyhedral algorithm for construction of tubes

In this Appendix we consider the problem of section 4, and we give an algorithm for construction of the effective and modified target sets $\mathbf{E}_N^*, \ldots, \mathbf{E}_1^*$, $\mathbf{X}_{N-1}^*, \ldots, \mathbf{X}_1^*$, when the sets $\mathbf{X}_k$, $\mathbf{U}_k$ are closed convex polyhedra, or unions of closed disjoint convex polyhedra, and the system is linear.

A polyhedron $\mathbf{P}$ in R^n is characterized by a finite set of vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$, the support set, and the values of its support functional

$$\sigma(\mathbf{e}_1|\mathbf{P}), \ldots, \sigma(\mathbf{e}_x|\mathbf{P})$$

at these vectors. It is the set of points x satisfying:

$$<\mathbf{x}, \mathbf{e}_i> \leq \sigma(\mathbf{e}_i|\mathbf{P}) \text{ for } i = 1, \ldots, k.$$

We give the following lemmas the proof of which can be found in Ref. [11].

Lemma A.1. Given a polyhedron $\mathbf{P}$ with support set $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$, and support functional

$$\sigma(\mathbf{e}_1|\mathbf{P}), \ldots, \sigma(\mathbf{e}_k|\mathbf{P})$$

the polyhedron $\mathbf{AP}$ (A: invertible matrix) has support set $\{\mathbf{A}'^{-1}\mathbf{e}_1, \ldots, \mathbf{A}'^{-1}\mathbf{e}_k\}$ and values of support functional

$$\sigma(\mathbf{A}'^{-1}\mathbf{e}_i|\mathbf{AP}) = \sigma(\mathbf{e}_i|\mathbf{P}), \qquad i = 1, \ldots, k.$$

Lemma A.2. Given two polyhedra $\mathbf{X}$ and $\mathbf{Y}$ with support sets $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$, $\{\mathbf{y}_1, \ldots, \mathbf{y}_m\}$, the vector sum $\mathbf{X}+\mathbf{Y}$ is a polyhedron with support set $\{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{y}_1, \ldots \mathbf{y}_m\}$ and support functional

$$\sigma(\mathbf{q}|\mathbf{X}+\mathbf{Y})=\sigma(\mathbf{q}|\mathbf{X})+\sigma(\mathbf{q}|\mathbf{Y})$$
$$\mathbf{q}=\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{y}_1, \ldots, \mathbf{y}_m.$$

Also, the intersection $X \cap Y$ is the polyhedron bounded by the hyperplanes

$$<\mathbf{x}, \mathbf{q}> \leq \min\{\sigma(\mathbf{q}|\mathbf{X}), \sigma(\mathbf{q}|\mathbf{Y})\}$$
$$\mathbf{q}=\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{y}_1, \ldots, \mathbf{y}_m.$$

We also prove the following Proposition:

Proposition A.1. If the polyhedron $\mathbf{X}_N$ of the target tube has support set $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ and support functional $\sigma(\mathbf{e}_1|\mathbf{X}_N), \ldots, \sigma(\mathbf{e}_k|\mathbf{X}_N)$ the effective target set $\mathbf{E}_N^*$ is the polyhedron bounded by the hyperplanes

$$<\mathbf{x}, \mathbf{e}_i> \leq \sigma(\mathbf{e}_i|\mathbf{X}_N)-\sigma(\mathbf{G}'_{N-1}\mathbf{e}_i|\mathbf{W}_{N-1})$$
$$i=1, \ldots, k \quad \text{(A.1)}$$

where $\sigma(\cdot|\mathbf{W}_{N-1})$ is the support functional of the set $\mathbf{W}_{N-1}$.

Proof. If $\mathbf{x} \in \mathbf{E}_N^*$ then

$$<\mathbf{x}, \mathbf{q}> + \sigma(\mathbf{G}'_{N-1}\mathbf{q}|\mathbf{W}_{N-1}) \leq \sigma(\mathbf{q}|X_N), \quad \mathbf{q} \in R^k$$

and $\mathbf{x} \in \mathbf{P}$ where $\mathbf{P}$ is the polyhedron bounded by the hyperplanes (A.1). Hence, $\mathbf{E}_N^* \subset \mathrm{P}$. Consider now the polyhedron $\mathbf{P}_w$ with support set $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ and support function

$$\sigma(\mathbf{e}_i|\mathbf{P}_w)=\sigma(\mathbf{G}'_{N-1}\mathbf{e}_i|\mathbf{W}_{N-1}),$$

$i=1, \ldots, k$. Then it is $\mathbf{G}_{N-1}\mathbf{W}_{N-1} \subset \mathbf{P}_w$. Using Lemma A.2 it is $\mathbf{P}+\mathbf{P}_w \subset \mathbf{X}_N$ and hence $\mathbf{G}_{N-1}\mathbf{W}_{N-1}+\mathbf{P} \subset \mathbf{X}_N$ which implies $\mathbf{P} \subset \mathbf{E}_N^*$. Hence $\mathbf{P}=\mathbf{E}_N^*$. Q.E.D.

We note that it is possible that not all of the hyperplanes (A.1) are support hyperplanes of $\mathbf{E}_N^*$ and before we proceed with the algorithm the redundant hyperplanes should be discarded using linear programming.

After the polyhedron $\mathbf{E}_N^*$ is determined, the modified target set $\mathbf{X}_{N-1}^*=\mathbf{X}_{N-1} \cap \mathbf{A}_{N-1}^{-1}[\mathbf{E}_N^*+(-\mathbf{B}_{N-1}\mathbf{U}_{N-1})]$ which clearly is a polyhedron, can be determined using Lemmas A.1, A.2 and linear programming. We proceed similarly to determine the remaining polyhedra of the tubes. It should be noted that the number of support hyperplanes of the polyhedra tends to increase as we go towards the initial time, and for high dimensional systems this way involve nontrivial storage requirements for the controller. On the other hand, the algorithm does not involve any approximations, and all computations are done off-line.

APPENDIX 2

An ellipsoidal approximation algorithm for construction of tubes

From the viewpoint of practically implementing the results of sections 2 through 4, it is clearly desirable that the effective and modified target sets be describable by a finite collection of numbers. Such is the case if, for example, these sets are ellipsoids. However, even if the system is linear and the various constraint sets are ellipsoids, these effective and modified target sets are not ellipsoids. On the other hand, a possible approach is to internally approximate these sets by ellipsoids, a procedure that not only allows us to easily implement the results of Sections 3 and 4 but, in addition, leads to control laws that are *linear*. It should be noted, however, that by internally approximating the true modified and effective target sets by ellipsoids the necessary and sufficient conditions obtained earlier become only sufficient.

Consider the special case of Problem 2 in which the system is linear and given by

$$\mathbf{x}_{k+1}=\mathbf{A}_k\mathbf{x}_k+\mathbf{B}_k\mathbf{u}_k+\mathbf{G}_k\mathbf{w}_k$$

and the relevant constraint sets are the ellipsoids described by

$$\mathbf{X}_k=\{z \in R^n: \ \mathbf{z}'\mathbf{C}'_k\mathbf{C}_k\mathbf{z} \leq 1\}$$
$$\mathbf{X}_N=\{z \in R^n: \ \mathbf{z}'\mathbf{\Psi}\mathbf{z} \leq 1\}$$
$$\mathbf{U}=\{\mathbf{v} \in R^m: \ \mathbf{v}'\mathbf{R}_k\mathbf{v} \leq 1\}$$
$$\mathbf{W}_k=\{\mathbf{v} \in \mathbf{R}^q: \ \mathbf{v}'\mathbf{D}_k\mathbf{v} \leq 1\}$$

and the matrices $\mathbf{\Psi}$, $\mathbf{R}_k$ and $\mathbf{D}_k$ are assumed positive definite for all $k=0, 1, \ldots, N-1$.

We first approximate the effective target set $\mathbf{E}_N^*$ by an ellipsoid. To this end, we state the following lemma, the proof of which can be found in Ref. [2].

Lemma A.3. Consider two ellipsoids $\mathbf{S}_1$, $\mathbf{S}_2$ with support functionals $\sigma(\mathbf{q}|\mathbf{S}_1)=(\mathbf{q}'\mathbf{Q}\mathbf{q})^{\frac{1}{2}}$, $\sigma(\mathbf{q}|\mathbf{S}_2)=(\mathbf{q}'\mathbf{Q}_2\mathbf{q})^{\frac{1}{2}}$. Their vector sum S_1+S_2 is contained in the ellipsoid S with support function $\sigma(\mathbf{q}|\mathrm{S})=\{\mathbf{q}'[\beta^{-1}\mathbf{Q}_1+(1-\beta)^{-1}\mathbf{Q}_2]\mathbf{q}\}^{\frac{1}{2}}$, where β is a free parameter with $0<\beta<1$.

We seek to internally approximate $\mathbf{E}_N^*$ by an ellipsoid $\hat{\mathbf{E}}_N^* \subset \mathbf{E}_N^*$. Hence we must have

$$\hat{\mathbf{E}}_N^*+\mathbf{G}_{N-1}\mathbf{W}_{N-1} \subset \mathbf{X}_N.$$

The support functionals of the ellipsoids $\mathbf{G}_{N-1}\mathbf{W}_{N-1}$ and $\mathbf{X}_N$ are,

$$\sigma(\mathbf{q}|\mathbf{G}_{N-1}\mathbf{W}_{N-1})=(\mathbf{q}'\mathbf{G}_{N-1}\mathbf{D}_{N-1}^{-1}\mathbf{G}'_{N-1}\mathbf{q})^{\frac{1}{2}}$$

and $\sigma(\mathbf{q}|\mathbf{X}_N)=(\mathbf{q}'\mathbf{\Psi}^{-1}\mathbf{q})^{\frac{1}{2}}$. By Lemma A.3 the relation $\hat{\mathbf{E}}_N^*+\mathbf{G}_{N-1}\mathbf{W}_{N-1} \subset \mathbf{X}_N$ is satisfied if the support functional of $\tilde{E}_N^*$ is given by $\sigma(\mathbf{q}|\tilde{\mathbf{E}}_N^*)=(\mathbf{q}'\mathbf{F}_N^{-1}\mathbf{q})^{\frac{1}{2}}$ where

$$\mathbf{F}_N^{-1}=(1-\beta_N)(\psi^{-1}-\beta_N^{-1}\mathbf{G}_{N-1}\mathbf{D}_{N-1}^{-1}\mathbf{G}'_{N-1}),\quad 0<\beta_N<1. \qquad (A.2)$$

If the given constraint sets are such that $\mathbf{E}_N^*$ has a nonempty interior, then there exists a β_N with $0<\beta_N<1$ such that the matrix $\mathbf{F}_N$ of (A.2) is positive definite and the ellipsoid

$$\tilde{\mathbf{E}}_N^*=\{\mathbf{z}:\ \mathbf{z}'\mathbf{F}_N\mathbf{z}\leqslant 1\} \qquad (A.3)$$

is contained in $\mathbf{E}_N^*$.

The modified target set is now defined, using the ellipsoid $\tilde{\mathbf{E}}_N^*$, as the set of points $\mathbf{x}_{N-1}$ with the property that both

$$\mathbf{x}'_{N-1}\mathbf{C}'_{N-1}\mathbf{C}_{N-1}\mathbf{x}_{N-1}\leq 1 \qquad (A.4)$$

and

$$\mathbf{x}_N=\mathbf{A}_{N-1}\mathbf{x}_{N-1}+\mathbf{B}_{N-1}\mathbf{u}_{N-1}\in\tilde{\mathbf{E}}_N^* \quad \text{for some } \mathbf{u}_{N-1}\in\mathbf{U}_{N-1}. \qquad (A.5)$$

The second requirement becomes in this case that

$$\mathbf{x}'_N\mathbf{F}_N\mathbf{x}_N\leq 1 \quad \text{for some } \mathbf{u}_{N-1} \text{ with } \mathbf{u}'_{N-1}\mathbf{R}_{N-1}\mathbf{u}_{N-1}\leq 1. \qquad (A.6)$$

The set of points satisfying both equations (A.4) and (A.6) clearly contains the set of points with the property that

$$\mathbf{x}'_{N-1}\mathbf{C}'_{N-1}\mathbf{C}_{N-1}\mathbf{x}_{N-1}+\mathbf{u}'_{N-1}\mathbf{R}_{N-1}\mathbf{u}_{N-1}+\mathbf{x}'_N\mathbf{F}_N\mathbf{x}_N\leq 1 \qquad (A.7)$$

where

$$\mathbf{x}_N=\mathbf{A}_{N-1}\mathbf{x}_{N-1}+\mathbf{B}_{N-1}\mathbf{u}_{N-1}. \qquad (A.8)$$

By well-known results on the linear quadratic problem of optimal control, see Ref. [9], the set of $\mathbf{x}_{N-1}$ satisfying equations (A.7) and (A.8) is given by

$$\tilde{\mathbf{X}}_{N-1}^*=\{\mathbf{x}_{N-1}:\ \mathbf{x}'_{N-1}K_{N-1}\mathbf{x}_{N-1}\leq 1\} \qquad (A.9)$$

where the positive definite matrix $\mathbf{K}_{N-1}$ is given by the discrete Riccati equation

$$\mathbf{K}_{N-1}=\mathbf{A}'_{N-1}[\mathbf{F}_N^{-1}+\mathbf{B}_{N-1}\mathbf{R}_{N-1}^{-1}\mathbf{B}'_{N-1}]^{-1}\mathbf{A}_{N-1}+\mathbf{C}'_{N-1}\mathbf{C}_{N-1}. \qquad (A.10)$$

Furthermore, a control law that achieves reachability is

$$\mathbf{u}_{N-1}(\mathbf{x}_{N-1})=-(\mathbf{R}_{N-1}+\mathbf{B}'_{N-1}\mathbf{F}_N\mathbf{B}_{N-1})^{-1}\mathbf{B}'_{N-1}\mathbf{F}_N\mathbf{A}_{N-1}\mathbf{x}_{N-1}. \qquad (A.11)$$

If $\tilde{\mathbf{X}}_{N-1}^*$ contains the set $\mathbf{G}_{N-2}\mathbf{W}_{N-2}$ the subsequent effective target set $\mathbf{E}_{N-1}^*$ is nonempty and we proceed with similar approximations. If some effective target set is empty, then the algorithm breaks down. This, of course, does not imply that the original target tube is not reachable, since the approximations make this condition sufficient only. If one wishes to proceed with the ellipsoidal algorithm he will have to start with a "larger" target tube. We summarize the algorithm below:

A suboptimal modified target tube $\{\tilde{\mathbf{X}}_1^*,\ldots,\tilde{\mathbf{X}}_N^*\}$ and effective target tube $\{\tilde{\mathbf{E}}_1^*,\ldots,\tilde{\mathbf{E}}_N^*\}$ are given recursively by:

$$\tilde{\mathbf{X}}_k^*=\{\mathbf{x}_k:\ \mathbf{x}'_k\mathbf{K}_k\mathbf{x}_k\leq 1\} \qquad k=1,\ldots,N$$

$$\tilde{\mathbf{E}}_k^*=\{\mathbf{x}_k:\ \mathbf{x}'_k\mathbf{F}_k\mathbf{x}_k\leq 1\} \qquad k=1,\ldots,N$$

where

$$\mathbf{F}_k^{-1}=(1-\beta_k)[\mathbf{K}_k^{-1}-\beta_k^{-1}\mathbf{G}_{k-1}\mathbf{D}_{k-1}^{-1}\mathbf{G}'_{k-1}]$$

$$\mathbf{K}_{k-1}=\mathbf{A}'_{k-1}[\mathbf{F}_k^{-1}+\mathbf{B}_{k-1}\mathbf{R}_{k-1}^{-1}\mathbf{B}'_{k-1}]^{-1}\mathbf{A}_{k-1}+\mathbf{C}'_{k-1}\mathbf{C}_{k-1}$$

$$\mathbf{K}_N=\psi$$

and the parameters β_k are such that $0<\beta_k<1$ and the matrices F_k are positive definite. A sufficient condition for reachability is then that the set

$$\tilde{\mathbf{T}}_o^*=\{\mathbf{x}_o:\ \mathbf{x}'_o\mathbf{K}_o\mathbf{x}_o\leq 1\}$$

contains $\mathbf{X}_o$, where

$$\mathbf{K}_o=\mathbf{A}'_o[\mathbf{F}_1^{-1}+\mathbf{B}_o\mathbf{R}^{-1}\mathbf{B}']^{-1}\mathbf{A}_o.$$

Furthermore, a control law that achieves reachability is given by:

$$\mathbf{u}_k(\mathbf{x}_k)=-(\mathbf{R}_k+\mathbf{B}'_k\mathbf{F}_{k+1}\mathbf{B}_k)^{-1}\mathbf{B}'_k\mathbf{F}_{k+1}\mathbf{A}_k\mathbf{x}_k. \qquad (A.12)$$

We remark that another control law that achieves reachability is the control law with a dead zone given by equation (A.12) when $\mathbf{x}'_k\mathbf{A}'_k\mathbf{F}_{k+1}\mathbf{A}_k\mathbf{x}_k>1$ (i.e. $\mathbf{A}_k\mathbf{x}_k\notin\tilde{\mathbf{E}}_{k+1}^*$) and $\mathbf{u}_k(\mathbf{x}_k)=0$ otherwise. In certain applications the use of a dead zone can be particularly beneficial.

Consider now the case where the system is constant (time-invariant) and the given constraint sets are constant. Suppose that the algebraic matrix equation

$$\mathbf{K}=\mathbf{A}'[(1-\beta)\mathbf{K}^{-1}-\beta^{-1}(1-\beta)\mathbf{G}\mathbf{D}^{-1}\mathbf{G}'+\mathbf{B}\mathbf{R}^{-1}\mathbf{B}']^{-1}\mathbf{A}+\mathbf{C}'\mathbf{C}$$

has a positive definite solution $\overline{\mathbf{K}}$ for some $0<\beta<1$ for which the matrix

$$_{-1}\mathbf{F}=(1-\beta)[\overline{\mathbf{K}}^{-1}-\beta^{-1}\mathbf{G}\mathbf{D}^{-1}\mathbf{G}']$$

is also positive definite. Then if the initial state belongs to the set $\mathbf{X}^* = \{\mathbf{x}:\ \mathbf{x}'\mathbf{K}\mathbf{x} \le 1\}$, then the state of the system can be made to stay indefinitely in the tube $\{\mathbf{X}^*, \mathbf{X}^*, \ldots\}$ and since $\mathbf{X}^* \subset \mathbf{X} = \{\mathbf{x} : \mathbf{x}'\mathbf{C}'\mathbf{C}\mathbf{x} \le 1\}$ infinite time reachability is achieved. The corresponding linear time-invariant control law that achieves reachability is

$$\mathbf{u}(\mathbf{x}) = -(\mathbf{R} + \mathbf{B}'\mathbf{F}\mathbf{B})^{-1}\mathbf{B}'\mathbf{F}\mathbf{A}\mathbf{x}$$

and it can have a dead zone if desirable.

The ellipsoidal algorithm presented in this Appendix has the drawback that the approximations involved may cause failure of existence of a solution even when an optimal solution exists. For this reason specification of "larger" target tubes and "larger" control sets may be necessary if a solution is to be achieved. Thus the procedure is not entirely satisfying. However, in view of the appeal of the linear control laws, it may prove useful in at least some practical cases. Also an important question that requires further consideration concerns the quality of the approximations involved in the algorithm. Unfortunately, it appears to be difficult to obtain precise estimates of the approximation involved and further research and simulations are required for a more complete evaluation of the merits and drawbacks of the algorithm.

Résumé—Cet article se rapporte à la commande en boucle fermée de systèmes à temps discret en présence d'incertitudes. L'incertitude peut avoir lieu sous la forme de perturbations dans la dyanmique du système, sous la forme de perturbations faussant les mesures à la sortie ou sous la forme d'une connaissance incomplète de l'état initial du système. Dans tous les cas, les grandeurs incertaines sont supposées inconnues sauf leur appartenance à des séries données. L'article considère d'abord le problème d'amener l'état du système au moment final dans une série préscrite de buts sous la plus mauvaise combinaison de perturbations. Ceci est ensuite généralisé au problème de maintenir toute la trajectoire de l'état à l'intérieur d'une "enveloppe" donnée de buts. L'article donne des conditions nécessaires et suffisantes pour la capacité d'atteindre une série de buts et une enveloppe de buts dans le cas où l'état peut être mesuré exactement, tandis que des conditions suffisantes pour cette capacité d'atteindre sont données lorsque seules des mesures à la sortie faussées par les perturbations sont disponibles. L'article donne un algorithme pour la construction efficace d'approximations elliptiques des séries en question et il est montré que cet algorithme conduit à des lois linéaires de commande. L'article discute également des applications de ces résultats à des jeux de poursuite-fuite.

Zusammenfassung—Die Arbeit befaßt sich mit der Regelung von diskontinuierlichen Systemen bei Vorhandensein einer Unbestimmtheit. Sie kann vorliegen in Form von Störungen in der Systemdynamik, von Störungen, die die Ausgangsmessungen fälschen oder von unvollständiger Kenntnis des Anfangszustandes des Systems. In allen Fällen werden die unbestimmten Größen als unbekannt, aber als in gegebenen Mengen liegend, angenommen. Betrachtet wird zunächst das Problem der Überführung des System-zustandes zur Endzeit in eine vorgeschriebene Zielmenge und zwar bei der ungünstigsten Kombination von Störungen. Dies wird auf das Problem der Beschränkung der ganzen Zustandstrajektorie auf einen gegebenen Ziel-"Schlauch" ausgedehnt. Notwendige und hinreichende Bedingungen werden für den Fall angegeben, daß der Systemzustand exakt gemessen werden kann, während hinreichende Bedingungen für die Erreichbarkeit für den Fall gegeben werden, wenn lediglich durch Störungen gefälschte Messungen vorhanden sind. Angegeben wird ein Algorithmus zur wirksamen Konstruktion von elliptischen Approximationen der enthaltenen Mengen. Weiter wird gezeigt, daß dieser Algorithmus zu linearen Regelungsgesetzen führt. Die Anwendung der hier gewonnnenen Ergebnisse auf eine Klasse von Verfolgungsspielen wird diskutiert.

Резюме—Настоящая статья относится к управлению в замкнутом контуре системами с дискретным временем в присутствии неопределенностей. Неопределенность может иметь место в смысле помех в динамике системы, в смысле помех искажающих выходные измерения или в смысле неполного знания начального состояния системы. Во всех случаях, неопределенные величины предполагаются неизвестными за исключением их принадлежности к определенным рядам. Статья рассматривает сначала задачу приведения состояния системы в конечный момент в заданный ряд целей при наихудшем сочетании помех. Это затем обобщается к задаче поддержания состояния внутри данной "оболочки" целей. Статья дает необходимые и достаточные условия для способности достижения ряда целей в случае когда состояние может быть точно измерено, в то время как достаточные условия для этой способности достижения даются когда имеются налицо лишь выходные измерения искаженные помехами. Статья дает алгоритм для эффективного построения эллиптических приближений к рассматриваемым рядам и показывает что этот алгоритм приводит к линейным законам управления. Статья также обсуждает применения своих результатов к играм преследования и побега.