

Identifying Single-Input Linear System Dynamics from Reachable Sets

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Abstract—This paper is concerned with identifying linear system dynamics without the knowledge of individual system trajectories, but from the knowledge of the system’s reachable sets observed at different times. Motivated by a scenario where the reachable sets are known from partially transparent manufacturer specifications or observations of the collective behavior of adversarial agents, we aim to utilize such sets to determine the unknown system’s dynamics. The paper has two contributions. Firstly, we show that the sequence of the system’s reachable sets can be used to uniquely determine the system’s dynamics for asymmetric input sets under some generic assumptions, regardless of the system’s dimensions. We also prove the same property holds up to a sign change for two-dimensional systems where the input set is symmetric around zero. Secondly, we present an algorithm to determine the system’s dynamics. We apply and verify the developed theory and algorithms on an unknown band-pass filter circuit solely provided the unknown system’s reachable sets over a finite observation period.

I. INTRODUCTION

This paper aims to determine whether it is possible to use a control system’s reachable sets obtained at different time instances to calculate the system’s dynamics. In certain instances, we may be able to determine an approximation of a system’s reachable sets over a finite observation period. The purpose of this paper is to show that such information can be utilized to arrive at a dynamic model for an unknown system. Practical applications may include system identification of high-density drone and missile swarms [1], [2] where the reachable set can be found by observing multiple agents collectively, but without the capability of distinguishing them. Other applications include predicting macro-level population behaviors, e.g., determining how crowd behavior changes under social or economic events like the introduction of a new population or changes in the stock market [3]. We may also be able to model internal body functions on the cellular level [4], namely understanding how cells change their identity and behavior in living systems. We first show that reachable sets — under some technical assumptions — will uniquely determine an unknown system’s true dynamics. After uniqueness is proven, we develop a method to identify a linear model of an unknown system’s behavior using its reachable sets.

Previous research in system identification presents the most closely related contributions to the method presented in this paper. However, previous work on system identification

classically relies on frequency response techniques induced by randomized actuator inputs [5], [6]. More sophisticated system identification techniques involve neural networks [7]. Single-layer and multi-layer neural networks have also been applied with the use of parameter estimation algorithms using a single hidden layer [8] and H_∞ control-induced excitations for robust identification of system nonlinearities [9]. More recent work involves using recurrent neural networks [10], [11] with Long Short-Term Memory Units (LSTM) and fractional order neural networks (FONN) [12], [13] to identify and control dynamic systems. These methods, however, cannot be used unless one has access to a system’s actuators or individual trajectories. The significant difference of our novel method is that it does not require control of any actuators to model an unknown system nor observations of individual trajectories.

On a high level, the problem in this paper involves identifying the behaviors or capabilities of an observed system under limited information. While there exist other methods for adversarial behavior recognition, those works are focused on determining adversarial agent goals by matching actions of an agent against a plan library [14]–[16]. More recent work [17], [18] proposes using evolving fuzzy systems and artificial intelligence to adaptively predict agent behavior. In contrast, our method is starkly different since it is not primarily concerned with predicting adversarial behavior, but determining all possible actions of an adversary within a time horizon. Thus, instead of using a library of finite predetermined adversarial actions, our method uses reachable sets to produce a dynamic model of an unknown system.

A. Notation

We denote the set of all $n \times m$ real and complex matrices by $\mathbb{R}^{n \times m}$ and $\mathbb{C}^{n \times m}$ respectively; for $M \in \mathbb{R}^{n \times m}$, we let $M^T \in \mathbb{R}^{m \times n}$ denote its transpose. Vectors e_1, \dots, e_n will denote the canonical basis vectors in \mathbb{R}^n . We let \mathbb{N} denote the set of all natural numbers, $\mathbb{Z}_{\geq 0}$ denote the set of non-negative integers, and $GL(n)$ denote the set of invertible square matrices of dimension $n \in \mathbb{N}$. Let \mathcal{S} be a set of points in \mathbb{R}^n . Then $\text{Conv}(\mathcal{S})$ denotes the convex hull of \mathcal{S} . Notation $B\mathcal{X}$ where $B \in \mathbb{R}^{n \times m}$ and $\mathcal{X} \subset \mathbb{R}^m$ denotes the set $B\mathcal{X} = \{Bx \mid x \in \mathcal{X}\}$. Given two sets $\mathcal{A}, \mathcal{B} \in \mathbb{R}^n$, we denote $\mathcal{A} \oplus \mathcal{B} = \{a+b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ as their Minkowski sum. Similarly, $\mathcal{A} \ominus \mathcal{B} = \{c \in \mathbb{R}^n \mid c \oplus \mathcal{B} \subseteq \mathcal{A}\}$ denotes the Minkowski difference. We also define $\mathcal{A} + b = \{a+b \mid a \in \mathbb{R}^n\}$ as the translation of \mathcal{A} by $b \in \mathbb{R}^n$.

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II. PROBLEM STATEMENT

We consider the discrete-time, single-input linear system

$$x[i+1] = Ax[i] + bu[i], \quad x[0] = 0, \quad (1)$$

where all $i \in \mathbb{Z}_{\geq 0}$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $u \in \mathcal{U} \subset \mathbb{R}$ where $\mathcal{U} = [\underline{u}, \bar{u}]$ such that $\underline{u} \neq \bar{u}$. We assume $b \neq 0$ since the system's reachable sets are trivial otherwise. We also assume $x[0] = 0$; by a shift in coordinates, the case of $x[0] \neq 0$ is equivalent to that of an *affine* system $x[i+1] = Ax[i] + bu[i] + c$ with initial state at the origin. Solving the problem in this setting can likely be approached by reproducing similar calculations in subsequent sections, but we leave such an effort for future work.

Our goal is to establish whether the dynamics of (1), i.e., matrices A and b , can be determined using the system's reachable sets. We now formally define said reachable sets.

Definition 1: For $i \in \mathbb{Z}_{\geq 0}$, the (forward) reachable set of system (1) at time i is

$$\mathcal{R}(i, x[0]) = \{\phi_u(i; x[0]) \mid u : \mathbb{Z}_{\geq 0} \rightarrow \mathcal{U}\},$$

where $\phi_u(\cdot; x[0])$ denotes the controlled trajectory of system (1) with control signal u .

We present the problem of whether the system dynamics are uniquely determined by the system's reachable sets.

Problem 1: Given a sequence of sets $\{\mathcal{R}(i, 0)\}_{i \in \mathbb{N}}$ which is generated by (1) for some (A, b) , determine whether (A, b) can be uniquely recovered from $\{\mathcal{R}(i, 0)\}_{i \in \mathbb{N}}$.

Notice that we explicitly assume the knowledge of all reachable sets at all times. Such an assumption might not always be realistic. We will show that we often need only the first $n+1$ reachable sets to uniquely recover the dynamics. We leave the more general case — where reachable sets at only some time steps are available — for future work.

The first step to solving Problem 1 is to derive a simple relationship between the system matrices and $\mathcal{R}(i, 0)$. Given system (1), we naturally utilize Minkowski sums and the Minkowski difference [19] to produce such a relationship for all $i \in \mathbb{N}$.

Theorem 1: Let $\mathcal{R}(i, 0)$ be the reachable set at time i of (1). Then

$$A^{i-1}b\mathcal{U} = \mathcal{R}(i, 0) \ominus \mathcal{R}(i-1, 0). \quad (2)$$

Proof: By (1) it is clear that $\mathcal{R}(1, 0) = b\mathcal{U}$. Since

$$x[i] = A^i x[0] + A^{i-1}bu[0] + \dots + bu[i-1],$$

clearly

$$\mathcal{R}(i, 0) = A^{i-1}b\mathcal{U} \oplus \mathcal{R}(i-1, 0).$$

We recall that the Minkowski sum of two convex sets is also convex [20]. Since all sets $A^{i-1}b\mathcal{U}$ are convex by the definition of \mathcal{U} , all sets $\mathcal{R}(i, 0)$ are convex by induction. Hence, the appropriate Minkowski difference [21] can be calculated to arrive at (2). ■

Theorem 1 implies that we can obtain $\{A^{i-1}b\mathcal{U}\}_{i \in \mathbb{N}}$ using the reachable sets $\mathcal{R}(i, 0)$. We will prove that when $\mathcal{U} \neq [-c, c]$ with $c \in \mathbb{R}$, matrices A and b are indeed

generically uniquely defined from $\{A^{i-1}b\mathcal{U}\}_{i \in \mathbb{N}}$, that is, uniquely defined under the assumptions formally written in Theorem 2 shown to be generic in a topological sense in Lemma 1. When $\mathcal{U} = [-c, c]$ for some $c \in \mathbb{R}$, we can show that (A, b) are not uniquely defined, but conjecture that they are unique up to a change in sign. We prove that this property holds for $n = 2$. We shall refer to solutions for cases with such a set \mathcal{U} as \pm -*unique*, which is explicitly defined in the next section.

Following Problem 1, which seeks to determine whether system dynamics are uniquely defined from reachable sets, we present the second problem, which aims to explicitly determine such dynamics.

Problem 2: Develop a method to recover at least one pair (A, b) which generates $\{\mathcal{R}(i, 0)\}_{i \in \mathbb{N}}$.

Based on methods in [19] for calculating Minkowski differences, we can calculate $\{A^{i-1}b\mathcal{U}\}_{i \in \mathbb{N}}$. We show in Section IV that the results of these Minkowski differences and knowledge of \mathcal{U} are sufficient for calculating $\{A^{i-1}b\}_{i \in \mathbb{N}}$, which in turn can be utilized to calculate the matrix pair (A, b) for controllable systems. We first tackle Problem 1.

III. UNIQUENESS OF THE DERIVED SYSTEM MODEL

We wish to determine when a pair (A, b) is uniquely defined by the reachable sets of (1). It can be easily shown that the answer is generally negative. Consider an unknown system (1) where $A = I$ and $b = [0 \ 1]^T$. Consider another system where the top left element of A equals 0; call this matrix A' . By equation (2) of Theorem 1, the reachable sets of (1) with matrix pairs (A, b) and (A', b) are the same. Thus, we begin by determining sufficient conditions which guarantee whether (A, b) can be uniquely recovered as stated in Problem 1. We will show uniqueness under several technical assumptions; Lemma 1 shows that these assumptions are generic in a topological sense. Its proof is available in this paper's extended version¹.

Lemma 1: Let $\mathcal{N} \subset \mathbb{R}^{n \times n}$ be the set of all matrices such that if $A \in \mathcal{N}$, then A^2 has all distinct eigenvalues. Let $b \in \mathbb{R}^n \setminus \{0\}$ and $\mathcal{O} \in \mathbb{R}^{n \times n}$ be the set of all matrices such that, if $A \in \mathcal{O}$ and $\eta \in \mathbb{C}^n$ is any left eigenvector of A , $b^T \eta \neq 0$. Then, $GL(n) \cap \mathcal{N} \cap \mathcal{O}$ is an open and dense set.

We emphasize that many well-known linear controllable systems, such as the discrete double integrator, RLC circuit, and linearized pendulum [22], contain A matrices which satisfy the conditions of Lemma 1. Also, these generic assumptions are not necessary, but sufficient to guarantee uniqueness. In the proof below, we will use the assumptions in Lemma 1 to prove that the dynamics derived from reachable sets are generically unique, at least for an asymmetric input set.

Theorem 2: Let $\mathcal{U} = [c, d]$, where $c \neq \pm d$. Let $\eta_i \in \mathbb{C}^n$ for $i \in \{1, \dots, n\}$ be the left eigenvectors of A . Let the sequence $\{\mathcal{R}(j, 0)\}_{j \in \mathbb{N}}$ be generated by system (1) for system matrices (A, b) and (A', b') , where $A, A' \in GL(n)$,

¹<https://arxiv.org/abs/2309.04340>

A and A' have n distinct eigenvalues, and $b^T \eta_i \neq 0$ for all i . Then, $(A, b) = (A', b')$.

Proof: If (A, b) and (A', b') for system (1) produce an identical sequence $\{\mathcal{R}(j, 0)\}_{j \in \mathbb{N}}$, then $\mathcal{R}(1, 0) = b\mathcal{U} = b'\mathcal{U}$, i.e., there are two options: (i) $bc = b'c$ and $bd = b'd$ or (ii) $bc = b'd$ and $bd = b'c$. If the latter option is true, then $bcd = b'd^2 = b'c^2$, so $b' = 0$. In that case, $b = 0$, so $b = b'$. If the former option is true, because at least one of c or d is non-zero, again $b = b'$.

Let us now perform a coordinate transformation $z = Mx$, where M is chosen so that $Mb = e_1$. Such an M exists since $b \neq 0$. Then, $\dot{z} = MAx + Mbu = MAM^{-1}z + e_1u$. If we define $\hat{A} = MAM^{-1}$, by our assumptions \hat{A} is invertible and \hat{A} has distinct eigenvalues. Additionally, it is trivially verified that left eigenvectors of \hat{A} are $(M^{-1})^T \eta_i$. Since $M^{-1}e_1 = b$, the assumption $b^T \eta_i \neq 0$ is equivalent to the first element of the left eigenvectors of \hat{A} being non-zero. To simplify the notation, by a standard abuse of notation we now let (A, e_1) , (A', e_1) denote the system matrices after performing the above transformation. By the above discussion, we are assuming that A and A' are invertible, have distinct eigenvalues, and that $\eta_{i1} \neq 0$ for all i .

Noting that the two systems produce the same reachable sets, by (2) it follows that $A^k e_1 = A'^k e_1$ for all $k \in \mathbb{N}$. By the same logic as in the first paragraph of the proof, we see that since $c \neq -d$, then $A^k c e_1 = A'^k c e_1$ and $A^k d e_1 = A'^k d e_1$ is satisfied for all $k \in \mathbb{N}$, giving us the relation

$$A^k e_1 = A'^k e_1 \quad \forall k \in \mathbb{Z}_{\geq 0}. \quad (3)$$

Equation (3) implies $A^{k-1} A' e_1 = A'^{k-1} A e_1$ and $A'^{k-2} A e_1 = A'^{k-1} e_1$ for all $k \geq 2$. We have

$$A^{k-1} A' e_1 = A'^{k-1} A e_1 = A' A'^{k-2} A e_1 = A' A'^{k-1} e_1.$$

Hence, $A^{k-1} A' e_1 = A' A'^{k-1} e_1$; since A' is invertible,

$$A^k e_1 = A'^{(-1)} A^k A' e_1 \quad \forall k \in \mathbb{Z}_{\geq 0}. \quad (4)$$

Let v_i denote the right eigenvectors of A and v'_i, η'_i denote the right and left eigenvectors of $A'^{(-1)} A A'$ respectively. Since A and $A'^{(-1)} A A'$ are similar matrices, their eigenvalues are equal [23]. Let $A = V D V^{-1}$ and $A'^{(-1)} A A' = V' D V'^{-1}$ where the rows of V^{-1} and V'^{-1} are η_i^T and $\eta'_i{}^T$ respectively and the columns of V and V' are v_i and v'_i respectively. By our assumptions, $\eta_{i1} \neq 0$, so we can now scale the η 's so that $\eta_{i1} = 1$. We then redefine v_i to be the newly scaled right eigenvectors such that $\eta_{i1} = 1$. Next, we write (4) in tensor notation [23] and subtract the right hand side from both sides to get

$$\sum_i \lambda_i^k (v_i - v'_i \eta'_{i1}{}^T) = 0 \quad \forall k \in \mathbb{Z}_{\geq 0}. \quad (5)$$

Taking $k \in \{0, \dots, n-1\}$ we have a series of n equations.

For the j -th element of any v_i and v'_i , we have

$$\Lambda S_j = \begin{bmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_n \\ \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \dots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} v_{1j} - v'_{1j} \eta'_{11}{}^T \\ v_{2j} - v'_{2j} \eta'_{21}{}^T \\ \vdots \\ v_{nj} - v'_{nj} \eta'_{n1}{}^T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

for any $j \in \{1, \dots, n\}$. Notice that $\Lambda \in \mathbb{C}^{n \times n}$ is the square Vandermonde matrix [24]. Recall that the Vandermonde matrix is invertible if elements λ_i are distinct for all i , which holds by assumption. If $\eta'_{i1} = 0$ for any i , then $v_i = 0$, which contradicts the assumption that A is diagonalizable. Consequently, $\eta'_{i1} \neq 0$ for all i , so similar to the previous step, we can scale v'_i and η'_{i1} such that $\eta'_{i1} = 1$ for all i . It follows that $v_{ij} = v'_{ij}$ for all i, j since Λ is invertible. Therefore, $A = A'^{(-1)} A A'$.

Recall that we assumed that all eigenvalues of A are distinct. Thus, since A and A' commute, we can conclude that A and A' have the same eigenvectors [25]. Recall that A and A' are both diagonalizable. If we take the eigenvalue expansion of A and A' and multiply both on the left by V^{-1} , then subtracting the right hand side from both sides of equation (3) implies

$$(\lambda_i^k - \lambda'_i{}^k) \eta_{i1} = 0 \quad \forall k \in \mathbb{N}.$$

By assumption, $\eta_{i1} \neq 0$ for all i , so $\lambda_i^k = \lambda'_i{}^k$ for all $k \in \mathbb{N}$. Therefore, both A and A' have the same eigenvectors and eigenvalues, hence $A = A'$. ■

Theorem 2 proves that given reachable sets of generic system (1), the pair (A, b) is uniquely defined when the set of control inputs is not symmetric around 0. We now want to address the degenerate case where $\mathcal{U} = [-c, c]$. It can be easily seen that in such a case, system (1) with (A, b) and $(-A, -b)$ will produce the same reachable sets. To discuss a relaxed notion of system uniqueness, we provide a formal definition of \pm -uniqueness.

Definition 2: The system dynamics (A, b) of (1) are \pm -unique if (A, b) and $(-A, -b)$ generate the same reachable sets, but there do not exist other pairs (A', b') which generate the same reachable sets.

We conjecture that in the case when \mathcal{U} is symmetric around 0 – a scenario common in many controls applications [26] – the dynamics are \pm -unique.

Conjecture 1: Let $\mathcal{U} = [-c, c]$. Let the sequence $\{\mathcal{R}(i, 0)\}_{i \in \mathbb{N}}$ be generated by (A, b) , where A^2 has distinct eigenvalues and (A, b) are known to satisfy the assumptions of Theorem 2. Then, (A, b) is \pm -unique.

Proving the conjecture above requires extensive theoretical developments and remains for future work. As an illustration, we formally prove the conjecture to be true in the two-dimensional case.

Theorem 3: Let $n = 2$. Then, Conjecture 1 is correct.

Proof: Similarly to the proof of Theorem 2, we have two options: $bc = b'c$ or $bc = -b'c$. In the former case, we reach the same result as before, namely $b = b'$. In the latter case, we obtain $b = -b'$. Altogether, we get $b = (-1)^{p(0)} b'$ where $p(0) \in \{0, 1\}$.

As in Theorem 2, through a coordinate transformation we assume without loss of generality that $b' = e_1$. Then $b\mathcal{U} = (-1)^{p(0)}b'\mathcal{U} = (-1)^{p(0)}[-c, c]e_1$. Following the same steps as in the beginning of the proof in Theorem 2, with a standard abuse of notation, we let A, A' represent the system dynamics in this new basis where A and A' satisfy our assumptions. Also, we find that if $\mathcal{U} = [-c, c]$, then we arrive at the relation

$$A^k e_1 = (-1)^{p(k)} A'^k e_1 \quad \forall k \in \mathbb{Z}_{\geq 0}. \quad (6)$$

When $k = 2$, we see that regardless of $p(1)$, $AA'e_1 = (-1)^{p(2)}A'Ae_1$. Using this fact along with equation (6) implies $A^{k-1}A'e_1 = (-1)^{p(k)}A'^{k-1}Ae_1$ for all $k \geq 1$ and $A^{k-2}A'e_1 = (-1)^{p(k-1)}A'^{k-2}Ae_1$ for all $k \geq 2$. We have

$$\begin{aligned} (-1)^{p(k)}A'^{k-1}Ae_1 &= (-1)^{p(k)}A'A'^{k-2}Ae_1 \\ &= (-1)^{p(k)}(-1)^{p(k-1)}(-1)^{p(1)}A'A^{k-1}e_1. \end{aligned}$$

Hence, $A^{k-1}A'e_1 = (-1)^{p(k)}(-1)^{p(k-1)}(-1)^{p(1)}A'A^{k-1}e_1$; since A' is invertible,

$$A^{k-1}e_1 = \frac{A'^{(-1)}A^{k-1}A'e_1}{(-1)^{p(k)}(-1)^{p(k-1)}(-1)^{p(1)}}.$$

We define $q(k) \in \{0, 1\}$ as follows: $(-1)^{q(k)} = \frac{(-1)^{p(k)}(-1)^{p(k-1)}(-1)^{p(1)}}{(-1)^{p(k)}(-1)^{p(k-1)}(-1)^{p(1)}}$. We then have

$$A^{k-1}e_1 = (-1)^{q(k)}A'^{(-1)}A^{k-1}A'e_1 \quad \forall k \in \mathbb{N}.$$

It holds that A^{k-1} and $A'^{(-1)}A^{k-1}A'$ have the same eigenvalues, so A^{k-1} and $-A'^{(-1)}A^{k-1}A'$ must have eigenvalues of opposite sign. That is, if λ_i and λ'_i are the eigenvalues of A and $\pm A'^{(-1)}AA'$ respectively, then $\lambda_i = \pm\lambda'_i$. Following the same steps as in the proof of Theorem 2 we get

$$\sum_i \lambda_i^{k-1} (v_i - (-1)^{q(k)} v'_i \eta'_{i1}{}^T) = 0 \quad \forall k \in \mathbb{N}. \quad (7)$$

We now show that if $(A, B) \in (\mathbb{R}^{2 \times 2}, \mathbb{R}^2)$, then equation (7) implies $A = \pm A'^{(-1)}AA'$. Recall that $q(k) \in \{0, 1\}$ and so $(q(1), q(2)) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. When $(q(1), q(2)) = (0, 0)$, equation (7) is the same as equation (5) for $k = 1$ and $k = 2$. If we write these equations in matrix form as in Theorem 2 we again have the Vandermonde matrix on the left-hand side. Following the same steps as Theorem 2, we see that $v_i = v'_i$ for all i . Since $\lambda_i = \pm\lambda'_i$, then $A = \pm A'^{(-1)}AA'$. Similarly, if $(q(1), q(2)) = (1, 1)$, if we follow the same procedure to find $A = \pm A'^{(-1)}AA'$.

The most interesting cases are when $(q(1), q(2)) \in \{(0, 1), (1, 0)\}$. Let us first consider $(q(1), q(2)) = (1, 0)$. Recall $q(k) \in \{0, 1\}$, so if $q(3) = 0$, then $(q(2), q(3)) = (0, 0)$. If $(q(k), q(k+1)) = (0, 0)$ for some k , we then have

$$\Lambda S_j = \begin{bmatrix} \lambda_1^k & \lambda_2^k \\ \lambda_1^{k+1} & \lambda_2^{k+1} \end{bmatrix} \begin{bmatrix} v_{1j} - v'_{1j} \eta'_{11}{}^T \\ v_{2j} - v'_{2j} \eta'_{21}{}^T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (8)$$

We note

$$\det(\Lambda) = \lambda_1^k \lambda_2^k \begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix} \neq 0$$

since we have two non-zero scalars multiplied by the non-zero Vandermonde determinant in the case of distinct eigenvalues. Hence, Λ as defined in (8) is invertible and we again conclude that $A = \pm A'^{(-1)}AA'$.

We lastly consider cases where $q(k)$ is alternating, namely $\{q(k)\}_{k=1}^3 = (0, 1, 0)$ and $\{q(k)\}_{k=1}^3 = (1, 0, 1)$. In the former case, we have

$$\Lambda S_j = \begin{bmatrix} 1 & 1 \\ \lambda_1^2 & \lambda_2^2 \end{bmatrix} \begin{bmatrix} v_{1j} - v'_{1j} \eta'_{11}{}^T \\ v_{2j} - v'_{2j} \eta'_{21}{}^T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The generic assumption that all eigenvalues are distinct modulo a sign implies that Λ is invertible, thus we again find $v_i = v'_i$ and thus $A = \pm A'^{(-1)}AA'$. By following the same steps, we arrive at the same conclusion when $\{q(k)\}_{k=1}^3 = (1, 0, 1)$.

We now have that $A = (-1)^{q(2)}A'^{(-1)}AA'$. If $q(2) = 0$, then A and A' commute. Using assumptions of the theorem statement, we can conclude that A and A' have the same eigenvectors [25]. If $q(2) = 1$, then $A = -A'^{(-1)}AA'$ and so $A^2 = A'^{(-1)}A^2A'$. Clearly, A^2 and A' commute, and by the theorem statement, A^2 has distinct eigenvalues, which again implies that A and A' share the same eigenvectors.

We now follow the same steps as in the latter part of the proof of Theorem 2. Namely, we can diagonalize A and A' ; taking the eigenvalue expansion of equation (6) and multiplying both sides on the left by the matrix of left eigenvectors gives us the series of equations

$$\lambda_i^{k-1} = (-1)^{q(k)} \lambda'_i{}^{k-1} \quad \forall k \in \mathbb{N}.$$

Since $q(2) = 0$ or $q(2) = 1$, then $\lambda_i = \lambda'_i$ or $\lambda_i = -\lambda'_i$ for all i . Since both A and A' have the same eigenvectors and eigenvalues same to a sign, then $A = \pm A'$. ■

Theorem 2 solves Problem 1 in the generic case where $\mathcal{U} \neq [-c, c]$ while Theorem 3 proves there exists a \pm -unique solution to Problem 1 in the two-dimensional case where $\mathcal{U} = [-c, c]$. The proof of Theorem 3 drives our intuition for Conjecture 1 in general: adding dimensions to the system *should not* make it more likely that multiple generic systems can produce the same reachable sets for all time, especially considering no two such systems exist when the input set is asymmetric. Formalizing this statement is left for future work.

IV. SOLVING FOR THE SYSTEM DYNAMICS

We ultimately want to use reachable sets to solve for the system dynamics. Equation (2) of Theorem 1 already gives us a formula for calculating $A^{i-1}b\mathcal{U}$ for all $i \in \mathbb{N}$, namely

$$A^{i-1}b\mathcal{U} = \mathcal{R}(i, 0) \ominus \mathcal{R}(i-1, 0).$$

In Theorem 2, we proved that the answer to Problem 1 is affirmative for generic, single-input linear systems, meaning that for cases where the linear system dynamics satisfy the generic assumptions of Lemma 1, we can uniquely determine the true dynamics from the system's reachable sets.

We will determine (A, b) from reachable sets through a two step procedure. First, we calculate $A^{i-1}b\mathcal{U}$ for

$i = \{1, \dots, n+1\}$. In the case where $\mathcal{U} \neq [-c, c]$, the sequence of sets $A^{i-1}b\mathcal{U}$ can be used to calculate (A, b) directly. If $\mathcal{U} = [-c, c]$, these same sets can be utilized to compute a number of candidate dynamics (A, b) which satisfy $\mathcal{R}(i, 0)$ for all i . To determine which candidate solutions are correct, we compute the forward reachable sets of (1) using all candidate (A, b) . By Theorem 3, in the two-dimensional case, only two solutions (A, b) and (A', b') such that $(A, b) = -(A', b')$ will satisfy $\mathcal{R}(i, 0)$ for all i .

We begin our method by first using an algorithm that takes reachable sets of (1) and solves for $A^{i-1}b\mathcal{U}$. By equation (2), we can utilize existing methods [19], [21], [27] to compute the Minkowski difference between two polygons to calculate $A^{i-1}b\mathcal{U}$ given $\mathcal{R}(i, 0)$ for all $i \in \mathbb{N}$. For this narrative, we adopt the method in [19]. By Lemma 1 of [19], if we let $v^{(i)} \in \mathcal{V}$ be the vertices of $\mathcal{R}(i-1, 0)$, then the Minkowski difference $\mathcal{R}(i, 0) \ominus \mathcal{R}(i-1, 0)$ may be computed by taking the intersection of the translation of the set $\mathcal{R}(i, 0)$ by vertices $v^{(i)} \in \mathcal{V}$ of $\mathcal{R}(i-1, 0)$:

$$\mathcal{R}(i, 0) \ominus \mathcal{R}(i-1, 0) = \bigcap_{v^{(i)} \in \mathcal{V}} (\mathcal{R}(i, 0) - v^{(i)}). \quad (9)$$

While computing the intersection in (9) is generally computationally difficult, calculations are made significantly easier as $A^{i-1}b\mathcal{U}$ is a line segment; see [19] for details.

We now move to recover $A^{i-1}b$ from $A^{i-1}b\mathcal{U}$. We consider two cases: $\mathcal{U} = [c, d]$ for $d \neq -c$ and $\mathcal{U} = [-c, c]$ for some $c \in \mathbb{R}$. In the former case, taking the mean of the vertices of $A^{i-1}b\mathcal{U}$ will provide $A^{i-1}b \frac{c+d}{2}$. Multiplying this vector by $\frac{2}{c+d}$ recovers $A^{i-1}b$, allowing us to identify A by the following theorem.

Theorem 4: Let us assume the n -dimensional system (1) is controllable. Let $C_{A,b} = [b \quad Ab \quad \dots \quad A^{n-1}b]$. For the single-input case, $A = AC_{A,b}C_{A,b}^{-1}$.

The proof of Theorem 4 is trivial, noting that $C_{A,b}$ is full rank for controllable systems. We note that the assumption of controllability is generic [26].

In the case where $\mathcal{U} = [-c, c]$, by multiplying the vertices of $A^{i-1}b\mathcal{U}$ by c , we can only recover $A^{i-1}b$ up to a sign, generating two candidates for each i . Substituting all possible candidates for $A^{i-1}b$ into the columns of $C_{A,b}$ and $AC_{A,b}$ generates 2^{n+1} candidate matrices A .

To determine which candidate solutions yield the correct \pm -unique matrix pair (A, b) , we can compute the reachable sets of all 2^{n+1} candidate solutions to solve for the desired unknown \pm -unique system dynamics. In the next section, we use the CORA toolkit [28] and adopt methods of computing the Minkowski difference detailed in [19] to numerically calculate the dynamics (A, b) for an unknown band-pass filter circuit system and a two-dimensional unknown system with $\mathcal{U} = [-1, 1]$, validating the developed theory.

V. NUMERICAL EXAMPLE

To validate the developed theory and demonstrate how to apply the proposed method, we first consider a scenario of reverse engineering an electric circuit from manufacturer

specifications. At times, manufacturers will only release partial information about a system. For example, instead of providing a dynamic model of a manufactured part, manufacturers might convey the set of all voltages a circuit may output within a set amount of time given the set of all viable input frequencies. Such information can be interpreted as the minimum time in which a state can be reached, providing a picture of the system's reachable sets. Motivated by such an example, in this section we provide an example of identifying the matrices (A, b) of a band-pass filter circuit from its reachable sets. This example utilizes the CORA toolkit [28] for set computations, namely to calculate convex hulls and Minkowski differences.

A. Band-Pass Filter Circuit

We present the linear dynamic model of a band-pass filter circuit [29]. Let us assume $x[0] = 0$. The state-space controllable canonical representation [26] of this circuit is

$$\begin{aligned} x[j+1] &= Ax[j] + bv_c[j] \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} x[j] + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v_c[j] \end{aligned} \quad (10)$$

such that $v_c[j] \in [0, 1]$ for all $j \in \mathbb{Z}_{\geq 0}$.

Assume the reachable sets $\{\mathcal{R}(j+1, 0)\}_{j=0}^{\infty}$ of the controllable dynamical system (10) are known. From controllability, we know the form of system (10), but not parameters a_0, a_1, a_2, a_3 . From this information, we want to recover the true parameters, which we set to $a_0 = 3$, $a_1 = 2$, $a_2 = 3$, and $a_3 = 6$. It can be easily shown that if $a_0 \neq 0$, the assumptions of Theorem 2 are satisfied. Clearly, a matrix M for which $Mb = e_1$ is a simple row permutation; the assumptions of Theorem 2 are invariant under permutations, hence all assumptions are satisfied and the results of Theorem 2 apply when solving for the matrix pair (A, b) . That is, there exists a unique matrix pair which satisfies $\{\mathcal{R}(j+1, 0)\}_{j=0}^{\infty}$. Since the system is four-dimensional, Theorem 4 shows we need only consider the sets $\{\mathcal{R}(j+1, 0)\}_{j=0}^4$ to calculate (A, b) .

Assume that $\{\mathcal{R}(j+1, 0)\}_{j=0}^4$ are known to equal

$$\begin{aligned} &\text{conv} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right), \text{conv} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right), \\ &\text{conv} \left(\begin{bmatrix} 0 \\ 0.86 \\ -6.02 \\ 33.00 \end{bmatrix}, \begin{bmatrix} 0 \\ -0.14 \\ 0.98 \\ -6.00 \end{bmatrix}, \begin{bmatrix} 0 \\ -0.14 \\ 0.98 \\ -5.00 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.86 \\ -6.02 \\ 34.00 \end{bmatrix} \right), \\ &\text{conv} \left(\begin{bmatrix} -0.15 \\ 1.05 \\ -5.99 \\ 33.00 \end{bmatrix}, \begin{bmatrix} 0.85 \\ -5.95 \\ 34.01 \\ -188 \end{bmatrix}, \begin{bmatrix} 0.85 \\ -5.95 \\ 34.01 \\ -187 \end{bmatrix}, \begin{bmatrix} -0.15 \\ 1.05 \\ -5.99 \\ 34.00 \end{bmatrix} \right), \text{ and} \\ &\text{conv} \left(\begin{bmatrix} -5.93 \\ 33.88 \\ -188.02 \\ 1035.00 \end{bmatrix}, \begin{bmatrix} 1.07 \\ -6.12 \\ 33.98 \\ -188.00 \end{bmatrix}, \begin{bmatrix} 1.07 \\ -6.12 \\ 33.98 \\ -187.00 \end{bmatrix}, \begin{bmatrix} -5.93 \\ 33.88 \\ -188.02 \\ 1036.00 \end{bmatrix} \right), \end{aligned}$$

respectively. Based on Theorem 2, the knowledge of only these five sets is sufficient to reconstruct the true values of parameters a_0, a_1, a_2 , and a_3 .

Since $\mathcal{R}(1,0) = b\mathcal{U}$ and $\mathcal{U} = [0,1]$, b can be trivially computed to equal $b = [0 \ 0 \ 0 \ 1]^T$. Using (2) where $\mathcal{U} = [0,1]$ we have $Ab = [0 \ 0 \ 1 \ -6]^T$, $A^2b = [0 \ 1 \ -6 \ 33]^T$, $A^3b = [1 \ -6 \ 33 \ -182]^T$, $A^4b = [-6 \ 33 \ -182 \ 1002]^T$. Recall we assume the system is controllable, and thus the controllability matrix $C_{A,b}$ is invertible. Finally, by Theorem 4, $A = [A^4b \ A^3b \ A^2b \ Ab] [A^3b \ A^2b \ Ab \ b]^{-1}$ which produces the correct matrix A , accurately reconstructing the parameters $a_0 = 3, a_1 = 2, a_2 = 3$, and $a_3 = 6$.

VI. CONCLUSION

This paper considers the problem of determining the dynamics of an unknown discrete-time linear system using its reachable sets. The theory developed in this paper proves that for input sets that are asymmetric around the origin, the derived system dynamics are, given some generic technical assumptions, unique. Thus, in such cases, we can determine the true dynamics of an unknown system using the sequence of the system's reachable sets. For the case where the input set is symmetric, we prove that the derived dynamics are unique up to a factor of ± 1 for two-dimensional systems and provide a conjecture that asserts the same result holds for n -dimensional systems. We develop a method for deriving the dynamics of a system given the sequence of the system's reachable sets using Minkowski differences and proceed to illustrate by example how the method can be applied to identify the unknown linear model of a band-pass filter.

A natural next step is to prove the stated conjecture to show \pm -uniqueness for n -dimensional systems. Also, our current technical assumptions are consistent with generic properties of matrices, but ideally we want to relax these assumptions to identify necessary conditions for uniqueness. We also want to consider cases when the state's initial conditions are non-zero, when there is only available knowledge of the system's reachable sets at non-consecutive time steps, and also when working with the more general framework of multi-input systems.

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