Guaranteed Reachability on Riemannian Manifolds for Unknown Nonlinear Systems

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Abstract

Determining the reachable set for a given nonlinear system is critically important for autonomous trajectory planning for reach-avoid applications and safety critical scenarios. Providing the reachable set is generally impossible when the dynamics are unknown, so we calculate underapproximations of such sets using local dynamics at a single point and bounds on the rate of change of the dynamics determined from known physical laws. Motivated by scenarios where an adverse event causes an abrupt change in the dynamics, we attempt to determine a provably reachable set of states without knowledge of the dynamics. This paper considers systems which are known to operate on a manifold. Underapproximations are calculated by utilizing the aforementioned knowledge to derive a guaranteed set of velocities on the tangent bundle of a complete Riemannian manifold that can be reached within a finite time horizon. We then interpret said set as a control system; the trajectories of this control system provide us with a guaranteed set of reachable states the unknown system can reach within a given time. The results are general enough to apply on systems that operate on any complete Riemannian manifold. To illustrate the practical implementation of our results, we apply our algorithm to a model of a pendulum operating on a sphere and a three-dimensional rotational system which lives on the abstract set of special orthogonal matrices.

Index terms— Reachable Set Computation, Nonlinear Control Systems, Uncertain Systems, Aerospace Systems, Autonomous Systems

I Introduction

Control algorithms have classically required models to manipulate system behavior, however to deal with scenarios where such models are not available, active controls research often

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aims to extend the capabilities of automation by developing sophisticated control algorithms under modeling uncertainties using some nominal representation of the dynamics. We go a step beyond to construct certifiably attainable control capabilities in real time for systems without knowledge of the system dynamics.

Work presented in [1, 2] sits within this general area, but those papers are unable to handle control systems that operate outside Euclidean space [3, 4, 5], which are becoming increasingly more important as state-of-the-art autonomous research involves adapting geometric methods for control of such systems [6, 7, 8]. We thus extend results to systems operating on general Riemannian manifolds [9, 10, 11] to incorporate this large class of systems in our novel theory. Motivated by examples such as a satellite becoming damaged in orbit [12], legged robotics applications [13], or other autonomous systems operating on manifolds experiencing abrupt changes in their dynamics [14, 15], we aim to underapproximate the unknown system's set of reachable states [16, 17] with limited information about the dynamics. We call such a set the *guaranteed reachable set* (GRS).

Unlike much of the previous work in reachable set analysis, the results presented in this paper focus on *guaranteed* reachability, i.e., states that are *provably* reachable, whereas much of previous work [18, 19] focuses on *overapproximations*, i.e., states that are *optimistically* reachable. We do so to develop the autonomous capability for a system to determine what it can provably achievable without knowledge of its dynamics. The primary contribution of this paper is to produce a meaningful underapproximation of the GRS for a nonlinear control-affine system operating on any complete Riemannian manifold. We do so without knowing almost anything about the system dynamics, only assuming knowledge of: (i) the local dynamics at some initial state possibly determined using persistent excitation [20], (ii) Lipschitz bounds on the growth rate of said local dynamics gathered using any prior knowledge of the system design and known physical laws, (iii) the Riemannian manifold and corresponding metric tensor on which the unknown system operates.

Previous reachability research under uncertainties requires significantly more information to implement than the method presented in this paper. Examples include computing reachable sets for dynamics with bounded disturbances [21, 22] or parametric uncertainties [23, 24]. More classical approaches of adaptive and robust control have applied similar methods without calculating reachable sets to achieve control objectives for uncertain systems on manifolds [25, 26], however, such methods assume more information regarding the system dynamics and lack certifiably achievable guarantees from reachability analysis. Previous work focusing on reachability analysis on manifolds [27, 28] also assumes complete or partial knowledge of the system dynamics. In contrast, our method requires no knowledge of the true dynamics of the unknown nonlinear control-affine system. Recent examples of data-driven framework implement neural network-based models to approximate reachable sets [29] and Monte Carlo methods to produce probabilistic reachable sets [30] but produce overapproximations and avoid producing any results in non-Euclidean space.

Our approach relies on the interpretation of the unknown nonlinear control system as an *ordinary differential inclusion* [31, 32, 33] (ODI) whose right-hand side is equal to the set of velocities the control system can achieve from any initial state on a Riemannian manifold. For an unknown system, exact velocities cannot be calculated from almost any state on a manifold, but we can determine a family of achievable velocity sets from any state using the local dynamics and Riemannian Lipschitz [34] bounds on the growth rate of

said dynamics. The intersection of all such sets forms the *guaranteed velocity set* (GVS). We underapproximate the GVS with a set characterized by a ball whose simple geometric properties allow for feasible real-time calculation.

The outline of this paper is as follows: we discuss the problem in greater detail in Section II, providing important definitions needed to understand results in subsequent sections. In Section III, we determine an underapproximation of the GVS as a subset of the tangent space for any existing point on a manifold. In Section IV, we determine how the GVS can be utilized to calculate the GRS. We then illustrate by example the theory working on two unknown systems, one operating on a one-dimensional sphere, and the other on $SO(3)$ in Section V.

Notation: Throughout this paper, we adopt the following notations. $\mathbb{R}^{n \times m}$: the set of all $n \times m$ real matrices; $[n]$: the set $\{1, \ldots, n\}$ where $n \in \mathbb{N}$; $N^T \in \mathbb{R}^{m \times n}$: the transpose of $N \in \mathbb{R}^{n \times m}$; $\lambda_i(N)$: the i-th eigenvalue of N; $\sigma_i(N)$: the i-th singular value of N; $N^\dagger$: the Moore-Penrose pseudoinverse of N; $\|\cdot\|$: the Euclidean norm on $\mathbb{R}^n$; $\|\cdot\|_1$: the one-norm on $\mathbb{R}^n$; $\mathbb{B}^n(a; b)$: a closed ball in $\mathbb{R}^n$ centered at $a \in \mathbb{R}^n$ with radius $b \geq 0$ under the Euclidean norm; $\text{Tr}(N)$: the trace $N \in \mathbb{R}^{n \times m}$; $\text{rank}(N)$: the rank of N; $\text{Im}(N)$: the image (range space) of N; $\text{Imm}(N)$: when matrix $Q = NP$ for some matrix P; $\text{Ker}(N)$: the kernel (null space) of N; (M^n, h): an n-dimensional Riemannian manifold with a Riemannian metric h; $\Gamma(M)$: the space of smooth vector fields on a manifold M; $T_x M$: the tangent space at $x \in M$; TM: the tangent bundle of M; $\mathbb{F}_p^{n \times m}$: a tensor space at $p \in M$ of dimension $n \times m$ constructed from vectors in $T_p M$; $h_x(u, v)$: the Riemannian inner product of $u, v \in T_x M$; $|\cdot|_{h_x}$: the norm induced by the Riemannian inner product h_x; ∇: the Levi-Civita connection; $\bar{\nabla}$: the flat connection; $\mathcal{L}(c)$: the length of curve c; $d(x, y)$: the Riemannian distance between x and y; $\tau_p^q V$: the parallel transport of vectors $v_i \in \Gamma(M)$ where $V = \begin{bmatrix} v_1 & v_2 & \ldots \end{bmatrix}$ from p to q, i.e., $\begin{bmatrix} \tau_p^q v_1 & \tau_p^q v_2 & \ldots \end{bmatrix}$.

II Problem Statement

We consider the nonlinear control-affine system $\mathcal{M}(f, G)$ on a manifold M^n [35, 36, 37] of the form

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t)$$
$$= f(x(t)) + \sum_{l=1}^{m} g_l(x(t))u_l(t), \quad x(0) = x_0 \in M \tag{1}$$

where the state space (M^n, h) is a complete, connected real n-dimensional analytic manifold with the Riemannian metric h, $x : [0, \infty) \to M$, $f, g_l \in \Gamma(M)$ are globally Riemannian Lipschitz continuous with Riemannian Lipschitz constants $L_f \geq 0$, $L_{g_l} \geq 0$, and $u(t) \in \mathbb{B}^m(0; 1) = \mathcal{U}$. Riemannian Lipschitz functions are a generalized notion of standard Lipschitz functions extended to manifolds which will be formally defined in subsequent sections. We begin by formally defining our first assumption, which is largely consistent with Assumption 1 of [1] with the added generalization of operating on any complete Riemannian manifold.

Assumption 1. *System $\mathcal{M}(f, G)$ lies on a complete n-dimensional Riemannian manifold with a known coordinate system. We denote h as the corresponding Riemannian metric. Functions f and G are of the form $f(x) = Rr(x)$ and $G(x) = RH(x)$ where $R \in \mathbb{F}_{x_0}^m$ is constant with respect to the basis of $T_{x_0}M$ and $r : M \to \mathbb{R}^m$, $H : M \to \mathbb{R}^{m \times m}$ are functions such that $H(x_0)$ is invertible.*

Manifolds come equipped with smooth atlases $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in \mathcal{A}}$ comprised of charts (U_α, ψ_α) such that U_α are open, $\cup_\alpha U_\alpha = M$ and $\psi_\alpha : M \to \mathbb{R}^n$. For any coordinate system (x) which holds $p \in U_\alpha$, the n vectors $\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p$ form a basis for the n-dimensional tangent space T_pM [37]. Hence, the basis for the tangent space is dependent on the choice of coordinates for U_α and the point p, which we know by Assumption 1.

Assumption 1 implies $\text{Im}(f(x)) \subset \text{Im}(G(x))$ with the case of full actuation, i.e., $m = n$, corresponding to $R = I$. The Riemannian metric in Assumption 1 provides us with knowledge of the intrinsic geometry; as mentioned, we can use coordinates informed by the this knowledge to determine a basis for any tangent space T_xM. By Assumption 1, $R \in \mathbb{F}_{x_0}^m$ is a vector-valued function that remains constant with respect to the basis of $T_{x_0}M$. In general, if $x_0 \neq x$, then $T_{x_0}M \neq T_xM$. These tangent spaces may contain different bases, and their connection defines how the basis changes from $T_{x_0}M$ to T_xM. When we refer to R as constant, we imply it remains constant with respect to the connection which defines the change of basis from $T_{x_0}M$ to T_xM. We will formally define a connection on a manifold in subsequent sections. Similar to previous work [20, 1, 2], we also assume knowledge on the local dynamics and growth rate bounds of system (1).

Assumption 2. *The local dynamics $f(x_0)$ and $g_l(x_0)$ are known as well as Riemannian Lipschitz bounds L_f, $L_{g_l} \in \mathbb{R}_+$. We also assume knowledge of $\text{Im}(R)$ and $\text{Imm}(R)$. However, we do not need to know the value of R exactly.*

Through previous work [20], we can calculate $f(x_0)$, $g_l(x_0)$ with arbitrarily small error using historical trajectory data. Using known physical laws, we may also determine some bound on the rate of change of the dynamics as the system travels along some manifold; to quantify this bound, we need to go over some geometric preliminaries to define the notion of parallel transport and the covariant derivative [38, 9]. We then show how parallel transport is used to define the notion of Lipschitzness on curved manifolds.

II-A Preliminaries

We present an example of a pendulum operating on the manifold $\mathcal{S}^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$ in Figure 1 to illustrate how a control system operates on a non-Euclidean manifold and the steps we take to develop our novel method. There are a number of new factors we must take into account in comparison to the work in [1, 2] when systems were constrained to operating on $\mathbb{R}^n$.

First, the notion of distance is not as intuitive on general Riemannian manifolds. For example, in Fig. 1, if we were to take the Euclidean normed difference between any two points on $\mathcal{S}^1$, the result no longer lies on the manifold. Thus, we need to determine a notion of distance on the manifold such that the distance between two points is constrained on M. Second, the method in [1, 2] functions by determining invariant velocities, i.e., the
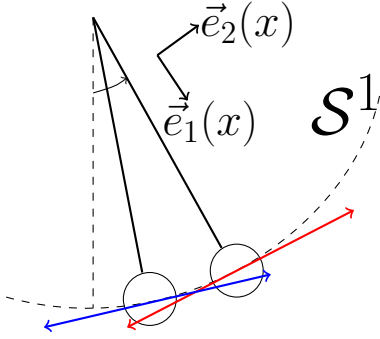
Figure 1: Pendulum constrained to $\mathcal{S}^1$ manifold with the tangent space $T_{x_0}M$ in blue and T_xM in red.

velocities that are available to the system at every x. However, if x lies on a non-Euclidean manifold, we need to find a way to identify velocities and tangent spaces at different states because $T_{x_0}M \neq T_xM$ in general cases. Lastly, after we find an underapproximation of our guaranteed velocities, we will need to determine what trajectories on M satisfy these velocities for some given time horizon.

We begin by addressing the first challenge, namely defining a notion of distance on M. To generalize the results from [1] to complete Riemannian manifolds, we must first define the Riemannian inner product — also referred to as the Riemannian metric — to develop a notion of distance on a manifold.

Definition 1 (Riemannian inner product). *Let M be a smooth manifold. Then, h is a smooth Riemannian inner product if for any $v_1, v_2 \in T_xM$, $h_x(v_1, v_2) = v_1^T H_x v_2$ where H_x is symmetric, positive definite and depends smoothly on $x \in M$.*

Following this definition, we see that the norm induced by the inner product on T_xM is then

$$|v|_{h_x} = \sqrt{h_x(v, v)} = \sqrt{v^T H_x v}. \tag{2}$$

We also denote H_x as the **Riemannian metric tensor**. In the right-hand side of (2), H_x is a symmetric, positive definite matrix at $x \in M$ whose representation depends on the basis vectors of T_xM. Note that as a consequence of positive definiteness, H_x is invertible [39]. In Euclidean space, $H_x = I$ and (2) becomes the Euclidean norm. Defining such a Riemannian metric h is always locally possible [9].

We now use the standard definition for vector induced norms on matrices [39] and the Riemannian metric to define the Riemannian matrix norm. Let $v \in \mathbb{R}^m$ and $a_i \in T_xM$ for all $i \in [m]$ such that $A = \begin{bmatrix} a_1 & a_2 & \dots & a_m \end{bmatrix}$. Then,

$$|A|_{h_x} = \sup_{v \neq 0} \frac{|Av|_{h_x}}{|v|_{h_x}}. \tag{3}$$

Using (2), (3) we calculate norm equivalence relations between the Riemannian and Euclidean norms for subsequent calculations.

Lemma 1. *Let v, a_i, A, and H_x be defined as above. Then*

$$\|H_x^{-1}\|^{-\frac{1}{2}}\|v\| \le |v|_{h_x} \le \|H_x\|^{\frac{1}{2}}\|v\| \tag{4}$$

and

$$\left(\frac{\|H_x^{-1}\|^{-1}}{\|H_x\|}\right)^{\frac{1}{2}}\|A\| \le |A|_{h_x} \le \|A\| \left(\frac{\|H_x\|}{\|H_x^{-1}\|^{-1}}\right)^{\frac{1}{2}}. \tag{5}$$

Proof. The matrix H_x is diagonalizable by the symmetry of H_x [39], so by (2) we have $|v|_{h_x}^2 = \sum_i \lambda_i(H_x)(v^i)^2 \le \lambda_{\max}(H_x)\|v\|^2$. For symmetric matrices, $\lambda_i(H_x) = \sigma_i(H_x)$ for all i [39], hence $\lambda_{\max}(H_x) = \|H_x\|$. Taking the square root of both sides gives us the right-hand side of (4). Repeating a similar process we see that $\lambda_{\min}(H_x)\|v\|^2 \le |v|_{h_x}^2$ implies the left-hand inequality in (4).

Equations (3) and (4) imply

$$|A|_{h_x} \le \sup_{v \ne 0} \frac{\|H_x\|^{\frac{1}{2}}\|Av\|}{\|H_x^{-1}\|^{-\frac{1}{2}}\|v\|} = \frac{\|H_x\|^{\frac{1}{2}}}{\|H_x^{-1}\|^{-\frac{1}{2}}} \sup_{v \ne 0} \frac{\|Av\|}{\|v\|}.$$

By definition of the Euclidean matrix norm [39] and the inequality above, we get the right-hand side of (5). Similar steps using the lower bound in (4) gives us the left-hand side of (5). □

To calculate an underapproximation of the reachable set, we will need to measure the length of a curve on a manifold. We begin by introducing the notion of a **simple curve**: a map $\gamma : [a,b] \to M$ is simple if it is piecewise differentiable and injective except on a finite subset of its domain [9]. The simple curve along with Riemannian metric can be used to define the Riemannian length functional and distance.

Definition 2 (Riemannian length functional). *Let $\gamma : [a,b] \to M$ be a simple curve. We define its length as*

$$\mathcal{L}(\gamma) := \int_a^b \sqrt{h_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt = \int_a^b |\dot{\gamma}(t)|_{h_{\gamma(t)}} \, dt.$$

The length $\mathcal{L}(\gamma)$ of two curves tracing the same path is agnostic under time reparameterizations [9]. The shortest length induced by the Riemannian metric is the Riemannian distance.

Definition 3 (Riemannian distance). *Denote $C(x,y)$ be the set of differentiable simple curves $\gamma : [a,b] \to M$ with $\gamma(a) = x$ and $\gamma(b) = y$. We define the Riemannian distance function $d : M \times M \to [0,\infty)$ between two points $x, y \in M$ by*

$$d(x,y) = \inf_{\gamma \in C(x,y)} \mathcal{L}(\gamma).$$

We can now use the notion of the Riemannian distance to define Lipschitz continuity on Riemannian manifolds. The corresponding Lipschitz constants can be used to determine maximal growth rate bounds for f, $g_l \in \Gamma(M)$. In subsequent sections, we use these maximal growth rate bounds on the dynamics of (1) to determine underapproximations of the guaranteed set of reachable states.

II-B   Lipschitz Continuity on Riemannian Manifold

Recall that in $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|f(x) - f(y)\| \le L\|x - y\|$$

for any $x, y \in \mathbb{R}^n$. However, since $T_xM \ne T_yM$, $f(x) - f(y)$ is undefined. We thus consider the Riemannian version of Lipschitz continuity found in Chapter II.A of [34]. We first need to define a map between tangent spaces. We begin by defining an **affine connection** [38, 9].

Definition 4 (Affine Connection). *An affine connection at x is a map*

$$\nabla : \ T_xM \times \Gamma(M) \to T_xM : \ (v, f) \to (\nabla_v f)(x)$$

with the properties:

1. *It is linear in v : $(\nabla_{av+bw} f) = a(\nabla_v f)(x) + b(\nabla_w f)(x)$.*

2. *It is additive in f : $\nabla_v(f + g) = (\nabla_v f)(x) + (\nabla_v g)(x)$.*

3. *It obeys the Leibniz (product) rule: for $\alpha : \ M \to \mathbb{R}$ a differentiable function, $\nabla_v(\alpha f) = (\partial_v \alpha)(x)f(x) + \alpha(x)(\nabla_v f)(x)$.*

Note ∂_v denotes the directional derivative. Let $\vec{e}_i(x) \in T_xM$ be basis vectors known by Assumption 1. Any connection that satisfies all properties of Definition 4 can be used to define the **covariant derivative** [40] of a vector field f along an integral curve $\gamma(t)$ in the direction $\dot{\gamma}(t) = \sum_{i=1}^n \dot{\gamma}^i \vec{e}_i$ with $\gamma(0) = x_0$:

$$\nabla_{\dot{\gamma}} f(x) := \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)).$$

We can write $f(x) = \sum_{j=1}^n f^j(x)\vec{e}_j(x)$ and apply the chain rule:

$$\nabla_{\dot{\gamma}} f(x) = \sum_{i,j=1}^n \dot{\gamma}^i \frac{\partial f^j(x)}{\partial x^i}\vec{e}_j + \sum_{j=1}^n f^j(x)\nabla_{\dot{\gamma}}\vec{e}_j. \tag{6}$$

Connection coefficients [38, 9] — also called Christoffel symbols — defined by $\nabla_{\vec{e}_i}\vec{e}_j = \sum_{k=1}^n \Gamma_{ij}^k \vec{e}_k$ quantify how the covariant derivative changes as the basis vectors of tangent planes on M rotate along a path on a curved surface. Using $\dot{\gamma} = \sum_{i=1}^n \dot{\gamma}^i \vec{e}_i$ and the additive and linear properties of Definition 4, (6) is equivalent to

$$\nabla_{\dot{\gamma}} f(x) = \sum_{i,j,k=1}^n \left(\dot{\gamma}^i \left(\frac{\partial f^k}{\partial x^i} + f^j \Gamma_{ij}^k \right) \right) \vec{e}_k. \tag{7}$$

We can define different connections by defining their corresponding connection coefficients Γ_{ij}^k. For the purposes of this paper, we will focus on two connections, namely the well-known Levi-Civita connection [38, 9] and the flat connection. The Levi-Civita connection is a torsion-free, metric-compatible connection commonly used in Riemannian geometry.

Formal definitions for **torsion-free** and **metric compatibility** can be found in [38, 9]. The corresponding connection coefficients are defined as

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l H^{kl} \left(\frac{\partial H_{li}}{\partial x^j} + \frac{\partial H_{lj}}{\partial x^i} - \frac{\partial H_{ij}}{\partial x^l} \right). \tag{8}$$

By Assumption 1, the Riemannian metric h is fully known and the knowledge of all coefficients follows from knowing h. In (8), H_{kl} and H^{kl} are the components of the Riemannian metric and inverse metric tensor respectively. In other words, H_{kl} and H^{kl} are components of H_x and H_x^{-1} respectively. With a connection formally defined, we can now define the **parallel transport** of a vector along a curve.

Definition 5 (Parallel Transport). *Let $\gamma : [0, 1] \to M$ be a differentiable curve in M and v_0 be a vector tangent to M at $\gamma(t_0)$. There exists a unique parallel vector field $V \in \Gamma(M)$ along γ, i.e., $\nabla_{\dot\gamma(t)} V(t) = 0$ such that $V(t_0) = v_0$. We call $V(t)$ the parallel transport of $V(t_0)$ along γ.*

From the definition above, the parallel transport along some curve γ is dependent on the defined connection. From the definition of metric compatibility, we can derive that the norm of a vector is invariant under the parallel transport defined using its corresponding connection [38, 9]. The norm induced by the Riemannian inner product is invariant under the parallel transport using the Levi-Civita connection. It follows by the definition of the matrix norm induced by the Riemannian metric defined in (3) that this norm is also invariant under parallel transport using the Levi-Civita connection.

As discussed previously, we will consider two connections, namely the Levi-Civita and flat connections. We already discussed the Levi-Civita connection for general Riemannian manifolds, so we now define the **flat connection** and identify its compatible metric. Let us denote $\tilde\Gamma_{ij}^k$ as the case when $\Gamma_{ij}^k = 0$ uniformly where Γ_{ij}^k is consistent with its definition in (8). Let us denote its corresponding affine connection as $\tilde\nabla$. By (7), this can be leveraged to define the covariant derivative using this connection. Using the covariant derivative with $\tilde\nabla$, we can define $\tilde\tau$ as the **flat transport** using Definition 5. Notice that $\Gamma_{ij}^k = 0$ uniformly when $H_x = I$ for all $x \in M$. By Definition 1, H_x defines some notion of a corresponding norm induced by the Riemannian inner product. Notice that for the case when $H_x = I$, then $|\cdot|_{h_x} = \|\cdot\|$. When the same metric tensor H_x used to define $|\cdot|_{h_x}$ is used to define Γ_{ij}^k in (7), we say the corresponding transport τ is metric compatible with $|\cdot|_{h_x}$. In the case where $H_x = I$, $|\cdot|_{h_x} = \|\cdot\|$ and $\Gamma_{ij}^k = \tilde\Gamma_{ij}^k = 0$, hence $\tilde\tau$ is compatible with $\|\cdot\|$, i.e., the flat transport is metric compatible with the Euclidean metric. This implies that for any $v \in T_x M$, $\tilde\tau_x^y v \in T_y M$, $\|v\| = \|\tilde\tau_x^y v\|$.

A **flat manifold** is defined as a manifold with zero curvature at every point, that is, when $\Gamma_{ij}^k = \tilde\Gamma_{ij}^k = 0$ for all $x \in M$. This implies that for flat manifolds, $H_x = I$ for all $x \in M$. Using equation (7), we see that the solutions to the system of ODEs $\tilde\nabla_{\dot\gamma} f(x) = 0$ are constants, which makes the calculation of parallel transport for such a connection trivial. An advantage of our results in the subsequent sections is that they are in terms of the flat connection and its corresponding transport. In such a case, the parallel transport calculation is trivial, which makes it ideal for real-time implementation. In general cases, solving the

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parallel transport in real time may be difficult because it involves determining a nontrivial solution to a system of ordinary differential equations at every time step.

Adopting the notation from Definition 5, Let $V(t)$ be the parallel transport of $V(t_0)$ along γ. To define Riemannian Lipschitz continuity of a vector $V(x) \in T_x M$ traveling to $V(y) \in T_y M$, we denote $\tau_x^y V := V(1) \in T_y M$.

Definition 6 (Classical Lipschitz Constant). *Let V be a continuous vector field on M. Then L is the classical Lipschitz constant on V if*

$$L = \sup_\gamma \frac{|\tau_\gamma V(\gamma(0)) - V(\gamma(1))|_{h_x}}{\mathcal{L}(\gamma)}$$

where $\gamma : [0,1] \to M$ varies over all $\mathcal{C}^1$-paths and τ_γ is shorthand for the parallel transport along the curve γ from $\gamma(0)$ to $\gamma(1)$.

This Lipschitz constant may seem difficult to calculate because it depends on varying over all $\mathcal{C}^1$-paths on M. We want to determine a more feasible way of knowing Lipschitz bound on a manifold to ensure that it is reasonable to have the knowledge from Assumption 2.

We begin by noting that for an arbitrary continuous vector field, L may be infinite, however if M is compact and if V is $\mathcal{C}^1$, then L is finite [34]. Recall that M is a complete Riemannian manifold; by the Hopf-Rinow Theorem, closed bounded sets of a connected Riemannian manifold are compact if and only if the manifold is complete [9, 38]. In subsequent sections, results will be obtained locally on closed, bounded subsets of M, thus we can assume L to be finite. We use compactness to apply the following lemma.

Lemma 2 (Lemma II.A.2.4, [34]). *Let $U \subseteq M$ be compact and $V : U \to TM$. Then, the supremum which determines the Lipschitz constant is attained if we vary only over geodesic paths in M.*

For a Riemannian manifold, there is a guaranteed existence of some geodesic curve $\gamma : [0,1] \to M$ connecting any two points $x, y \in M$ [9]. Additionally, for a complete manifold, given an appropriate neighborhood U, there exists some unique geodesic, so the Lipschitz constants need only be calculated for a single geodesic path. This implies Assumption 2 is reasonable since there is no need to vary over infinite $\mathcal{C}^1$ paths to find L_f and L_{g_l}. We can now express the Riemannian Lipschitz bounds introduced in Assumption 2 using the Riemannian metric $|\cdot|_{h_x}$ and Riemannian distance function $d(\cdot, \cdot)$. Let $U \subseteq M$ be compact, $V : U \to TM$, and assume system (1) is traveling along some path $\gamma : [0,1] \to V$ joining two points $p, q \in U$; if

$$|\tau_p^q V(p) - V(q)|_{h_x} \leq L d(p, q),$$

then any path γ joining p and q is L-path Lipschitz.

We know Riemannian Lipschitz bounds on the unknown dynamics, namely we are given the right-hand sides of inequalities

$$|\tau_{x_0}^x f(x_0) - f(x)|_{h_x} \leq L_f d(x_0, x)$$

and

$$|\tau_{x_0}^x g_l(x_0) - g_l(x)|_{h_x} \leq L_{g_l} d(x_0, x)$$

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for every $l \in [m]$. Given we work with a series of vectors transported along a Riemannian manifold, it would be helpful to characterize a bound on the set of vectors undergoing parallel transport simultaneously.

Lemma 3. *Let L_{g_l}, $g_l(x_0)$, n, and H_x be as defined above. Define $\overline{L}_g = \max_l\{L_{g_l}\}$. If $L_G = n\|H_x^{-1}\|\|H_x\|^{\frac{1}{2}}\overline{L}_g$, then*

$$|\tau_{x_0}^x G(x_0) - G(x)|_{h_x} \leq L_G d(x_0, x). \tag{9}$$

Proof. We begin by setting $l \in [m]$ to the index such that $L_{g_l} = \overline{L}_g$. By the equivalence relation (4), we have $\|H_x^{-1}\|^{-\frac{1}{2}}\|\tau_{x_0}^x g_l(x_0) - g_l(x)\| \leq |\tau_{x_0}^x g + l(x_0) - g_l(x)|_{h_x}$. Additionally, through vector norm equivalence relations [39], $\|\tau_{x_0}^x g_l(x_0) - g_l(x)\|_1 \leq \sqrt{n}\|\tau_{x_0}^x g_l(x_0) - g_l(x)\|$. Using the bound from Assumption 2, we can thus conclude that

$$\frac{\|H_x^{-1}\|^{-\frac{1}{2}}}{\sqrt{n}}\|\tau_{x_0}^x g_l(x_0) - g(x)\|_1 \leq \overline{L}_g d(x_0, x). \tag{10}$$

Notice that, by the definition of the induced matrix 1-norm [39], $\|\tau_{x_0}^x g_l(x_0) - g(x)\|_1 = \|\tau_{x_0}^x G(x_0) - G(x)\|_1$. By matrix norm equivalence relations [39], $\|\tau_{x_0}^x G(x_0) - G(x)\| \leq \sqrt{n}\|\tau_{x_0}^x G(x_0) - G(x)\|_1$. Using (5), we also get $(\|H_x\|\|H_x^{-1}\|)^{-\frac{1}{2}}|\tau_{x_0}^x G(x_0) - G(x)|_{h_x} \leq \|\tau_{x_0}^x G(x_0) - G(x)\|$. This inequality implies

$$\frac{(\|H_x\|\|H_x^{-1}\|)^{-\frac{1}{2}}}{\sqrt{n}}|\tau_{x_0}^x G(x_0) - G(x)|_{h_x} \leq \|\tau_{x_0}^x g_l(x_0) - g(x)\|_1.$$

Substituting the left-hand side of this inequality into the $\|\tau_{x_0}^x g_l(x_0) - g(x)\|_1$ term in (10) and multiplying both sides by $n\|H_x^{-1}\|\|H_x\|^{\frac{1}{2}}$ proves the claim. □

The lemma above leverages norm equivalence relations to get a Riemannian Lipschitz bound L_G on the matrix-valued function $G(x_0)$ composed of columns of vectors $g_l(x_0) \in T_{x_0}M$ for any $x_0 \in M$. From Lemma 3 and Assumption 2, we know $L_G \in \mathbb{R}_+$ and thus we know the right-hand side of the inequality

$$|\tau_{x_0}^x G(x_0) - G(x_0)|_{h_x} \leq L_G d(x_0, x).$$

Let us now denote a set $\Gamma_{L_f}(M)$ as the set of all $\hat{f} \in \Gamma(M)$ such that $|\tau_p^q f(p) - f(q)|_{h_x} \leq L_f d(p, q)$ for all $p, q \in U \subset M$. Next, we denote $\mathcal{D}_{con} \subseteq \Gamma_{L_f}(M) \times \Gamma_{L_G}(M)$ as the set of all Lipschitz $\hat{f}, \hat{g}_l \in \Gamma(M)$ that are consistent with Assumptions 1 and 2. We now utilize the set $\mathcal{D}_{con}$ to define the *guaranteed velocity set* (GVS).

II-C Guaranteed Velocity Set

We begin by following the approach of interpreting ordinary differential equations with control inputs as inclusions [31, 32, 33] as in previous work [1, 2]. We will later show how to integrate the sets on the right-hand side of the inclusion over a manifold to arrive at a set of guaranteed reachable states.

We adopt the same definitions for the *available velocity set* and *guaranteed velocity set* as in [1] with the only difference being how we defined $\mathcal{D}_{con}$ for systems operating on Riemannian manifolds in the previous section. We define the *available velocity set* of system (1) at state x by $\mathcal{V}_x = f(x) + G(x)\mathcal{U}$ and introduce the following ODI:

$$\dot{x} \in \mathcal{V}_x = f(x) + G(x)\mathcal{U}, \quad x(0) = x_0. \tag{11}$$

We now define the *guaranteed velocity set* (GVS) below.

Definition 7 (Guaranteed velocity set). *Let $(\hat{f}, \hat{G}) \in \mathcal{D}_{con}$ be consistent with knowledge from Assumption 2. Then, the guaranteed velocity set is*

$$\mathcal{V}_x^{\mathcal{G}} = \bigcap_{(\hat{f},\hat{G})\, \in\, \mathcal{D}_{con}} \hat{f}(x) + \hat{G}(x)\mathcal{U} \subseteq \mathcal{V}_x. \tag{12}$$

The GVS is the set of all velocities that can be taken by all systems given Assumption 2. Given Assumption 2, $\mathcal{V}_{x_0}^{\mathcal{G}}$ is known; using $\mathcal{V}_{x_0}$ and $\mathcal{D}_{con}$ we will develop guaranteed underapproximations for $\mathcal{V}_x^{\mathcal{G}}$.

Clearly, if a trajectory satisfies the inclusion $\dot{x} \in \mathcal{V}_x^{\mathcal{G}}$, it satisfies (11) and it serves as a solution to the control system (1) for an admissible control input. We utilize this fact to calculate a *guaranteed reachable set* from an underapproximation of $\mathcal{V}_x^{\mathcal{G}}$.

II-D   Guaranteed Reachable Set

We ultimately want to underapproximate the set of reachable states using solely the knowledge of $\mathcal{D}_{con}$. We first define the *(forward) reachable set*.

Definition 8 (Forward reachable set). *Let $\phi_u^{\hat{f},\hat{G}}(\cdot; x_0)$ denote the controlled trajectory of $\mathcal{M}(\hat{f}, \hat{G})$ with control signal u where $\phi_u^{\hat{f},\hat{G}}(0; x_0) = x_0$. Then, the (forward) reachable set is*

$$\mathcal{R}^{\hat{f},\hat{G}}(T, x_0) = \{\phi_u^{\hat{f},\hat{G}}(t; x_0) \mid u : [0, T] \to \mathcal{U}, t \in [0, T]\}.$$

Using this definition, we can easily define the *guaranteed reachable set* (GRS), which is simply the intersection of all forward reachable sets for all $(\hat{f}, \hat{G}) \in \mathcal{D}_{con}$ up to some time T.

Definition 9 (Guaranteed reachable set). *Let $T \geq 0$. We describe the guaranteed reachable set as:*

$$\mathcal{R}^{\mathcal{G}}(T, x_0) = \bigcap_{(\hat{f},\hat{G})\in\mathcal{D}_{con}} \mathcal{R}^{\hat{f},\hat{G}}(T, x_0). \tag{13}$$

With the limited knowledge from our assumptions, it is not possible to calculate this set in real time for general cases. This motivates the following problem statement.

Problem Statement 1. *Determine or underapproximate the GRS.*

The GRS can be calculated using the GVS [1, 2]. Recall that $\mathcal{V}_x^{\mathcal{G}}$ is the set of all velocities that can be taken by all systems with the assumed knowledge of $\mathcal{D}_{con}$. Let us consider the following ODI:

$$\dot{x} \in \mathcal{V}_x^{\mathcal{G}}, \quad x_0 = 0. \tag{14}$$

If $\mathcal{V}_{\phi(T;x_0)}^{\mathcal{G}} = \emptyset$, we will consider by convention that the trajectory of (14) ceases to exist at time T. The following proposition then holds directly from (12) and (13) and was directly shown in [41], just not on M.

Proposition 1. *Let $T \geq 0$. If a trajectory $\phi : [0, +\infty) \to M$ satisfies (14) at all times $t \leq T$, then $\phi(T) \in \mathcal{R}^{\mathcal{G}}(T, 0)$.*

Proposition 1 implies that the reachable set of (14) is a subset of $\mathcal{R}^{\mathcal{G}}(T, x_0)$. In the next section, we formulate an underapproximation of $\mathcal{V}_x^{\mathcal{G}}$. We then formulate a control system whose velocities are contained within these underapproximations in Section IV. The trajectories of this control system forms the reachable set. In the case where the tangent bundle does not lie on the same space as the manifold, an exponential mapping is applied to map guaranteed velocities to corresponding geodesics on M.

III   Guaranteed Velocity Set

In this section, we determine a guaranteed set of attainable velocities the unknown system can reach using knowledge consistent with our assumptions. We begin by determining the domain on x under which Assumption 1 remains satisfied, namely that $\text{Im}(\tilde{\tau}_{x_0}^x G(x_0)) = \text{Im}(G(x))$ using predefined coordinate bases and the flat transport $\tilde{\tau}$ between $T_{x_0}M$ and T_xM. It is clear from Assumption 1 that $\text{Im}(\tilde{\tau}_{x_0}^x G(x_0)) = \text{Im}(R)$. We want to show that for some neighborhood around x_0, $\text{Im}(G(x)) = \text{Im}(R)$ as well.

Lemma 4. *Set $g_l^\Gamma = \sum_{i,j,k} \dot{\gamma}^i \Gamma_{ij}^k g_l^j(x_0)\vec{e}_k$ for $l \in [m]$ where vectors $\vec{e}_k$ form the basis for T_xM. Under Assumptions 1 and 2, if*

$$d(x_0, x) < \frac{\|\tilde{\tau}_{x_0}^x G^\dagger(x_0)\|^{-1} - \|H_x^{-1}\|^{\frac{1}{2}}\|H_x\| \, \| \begin{bmatrix} g_1^\Gamma & \cdots & g_m^\Gamma \end{bmatrix} \|}{(\|H_x^{-1}\|\|H_x\|)^{\frac{1}{2}} L_G},$$

then $\text{Im}(\tilde{\tau}_{x_0}^x G(x_0)) = \text{Im}(G(x))$.

Proof. To prove the claim, we utilize Weyl's inequality [42] for singular values, which provides a Euclidean-normed bound on the perturbations of singular values. Consequently, we are motivated to determine an upper bound on $\|\tilde{\tau}_{x_0}^x G(x_0) - G(x)\|$ to exploit metric compatibility properties between the flat connection and its corresponding Euclidean norm. We note that $\|\tilde{\tau}_{x_0}^x G(x_0) - G(x)\| = \|\tilde{\tau}_{x_0}^x G(x_0) - \tau_{x_0}^x G(x_0) + \tau_{x_0}^x G(x_0) - G(x)\| \leq \|\tilde{\tau}_{x_0}^x G(x_0) - \tau_{x_0}^x G(x_0)\| + \|\tau_{x_0}^x G(x_0) - G(x)\|$, so any upper bound on $\|\tilde{\tau}_{x_0}^x G(x_0) - \tau_{x_0}^x G(x_0)\| + \|\tau_{x_0}^x G(x_0) - G(x)\|$ is also an upper bound on $\|\tilde{\tau}_{x_0}^x G(x_0) - G(x)\|$.

We begin by finding an upper bound on $\|\tau_{x_0}^x G(x_0) - G(x)\|$ using inequality (9) from Lemma 3 and substituting the left-hand side of inequality (5) into the left-hand side of (9). We now multiply both sides by $(\|H_x^{-1}\|\|H_x\|)^{\frac{1}{2}}$ to get

$$\|\tau_{x_0}^x G(x_0) - G(x)\| \le (\|H_x^{-1}\|\|H_x\|)^{\frac{1}{2}} L_G d(x_0, x). \tag{15}$$

Next, we need to quantify $|\tilde{\tau}_{x_0}^x G(x_0) - \tau_{x_0}^x G(x_0)|_{h_x}$ and use the norm equivalency relations from Lemma 1 to find an upper bound on $\|\tilde{\tau}_{x_0}^x G(x_0) - \tau_{x_0}^x G(x_0)\|$. By Definitions 4 and 5, we find that if we travel along some curve $\gamma : [0, 1] \to M$, the parallel transport provides a unique parallel vector field in terms of the covariant derivative corresponding to the appropriate connection. We want to measure the difference of these aforementioned parallel vector fields expressed in terms of the covariant derivative. Thus, we measure $|\tilde{\tau}_{x_0}^x G(x_0) - \tau_{x_0}^x G(x_0)|_{h_x}$ by quantifying $|\tilde{\nabla}_{\dot\gamma} G(x_0) - \nabla_{\dot\gamma} G(x_0)|_{h_x}$ where $\tilde{\nabla}, \nabla$ are the connections corresponding to $\tilde{\tau}$ and τ respectively. We then convert to the desired Euclidean metric to apply Weyl's inequality.

We begin by reminding the reader that $G(x_0)$ is defined $G(x_0) = \begin{bmatrix} g_1(x_0) & \cdots & g_m(x_0) \end{bmatrix}$ such that $g_l(x_0) \in T_{x_0} M$. In other words, $G(x_0)$ is treated as a collection matrix of vectors in $T_{x_0} M$, so to parallel transport $G(x_0)$ is to perform said transport $\tau_{x_0}^x g_l(x_0)$ for all $l \in [m]$. By extension, $\nabla_{\dot\gamma} G(x_0) = \begin{bmatrix} \nabla_{\dot\gamma} g_1(x_0) & \cdots & \nabla_{\dot\gamma} g_m(x_0) \end{bmatrix}$ is also defined component-wise for each component vector $g_l(x_0)$.

For any $l \in [m]$, using (7) and substituting $\tilde{\Gamma}_{ij}^k = 0$ for all i, j, k, we can calculate

$$\tilde{\nabla}_{\dot\gamma} g_l(x_0) - \nabla_{\dot\gamma} g_l(x_0) = \sum_{i,j,k} \dot\gamma^i \Gamma_{ij}^k(x) g_l^j(x_0) \vec{e}_k. \tag{16}$$

For notational simplicity, let g_l^Γ equal the right-hand side of (16) for any $l \in [m]$. The discussion above along with equation (16) implies that $\tilde{\nabla}_{\dot\gamma} G(x_0) - \nabla_{\dot\gamma} G(x_0) = \begin{bmatrix} g_1^\Gamma & \cdots & g_m^\Gamma \end{bmatrix}$, so from Definition 5 we can conclude

$$\left| \tilde{\tau}_{x_0}^x G(x_0) - \tau_{x_0}^x G(x_0) \right|_{h_x} = \left| \begin{bmatrix} g_1^\Gamma & \cdots & g_m^\Gamma \end{bmatrix} \right|_{h_x}. \tag{17}$$

The left and right-hand sides of (17) can be bounded by the left and right-hand sides of inequalities (4) and (5) respectively. Then we multiply both sides by $(\|H_x^{-1}\|\|H_x\|)^{\frac{1}{2}}$ to get

$$\|\tilde{\tau}_{x_0}^x G(x_0) - \tau_{x_0}^x G(x_0)\| \le \|H_x^{-1}\|^{\frac{1}{2}} \|H_x\| \left\| \begin{bmatrix} g_1^\Gamma & \cdots & g_m^\Gamma \end{bmatrix} \right\|. \tag{18}$$

From (15) and (18) it follows that

$$\|\tilde{\tau}_{x_0}^x G(x_0) - G(x)\| \le$$
$$(\|H_x^{-1}\|\|H_x\|)^{\frac{1}{2}} \left(\|H_x\|^{\frac{1}{2}} \left\| \begin{bmatrix} g_1^\Gamma & \cdots & g_m^\Gamma \end{bmatrix} \right\| + L_G d(x_0, x) \right). \tag{19}$$

For notational ease, let us denote $\eta(x_0, x)$ as the right hand side of (19). As discussed earlier, the flat transport is said to be metric compatible with the Euclidean norm, i.e., $\|G(x_0)\| = \|\tilde{\tau}_{x_0}^x G(x_0)\|$ because $\Gamma_{ij}^k = 0$ for the flat connection and its corresponding Euclidean metric. We can now apply Weyl's inequality, which states $\|\sigma_s(G(x)) - \sigma_s(\tilde{\tau}_{x_0}^x G(x_0))\| \le \|G(x) - \tilde{\tau}_{x_0}^x G(x_0)\| \le \eta(x_0, x)$ such that $1 \le s \le r$ with $r = \mathrm{rank}(\tilde{\tau}_{x_0}^x G(x_0))$. By the Eckhart-Young Mirsky theorem [39] and singular value decomposition, it holds that $\sigma_r(G(x_0)) = \|G^\dagger(x_0)\|^{-1} = \|\tilde{\tau}_{x_0}^x G^\dagger(x_0)\|^{-1}$; the second equality stems from the norm preservation property of compatible metrics. Note that $\|\tilde{\tau}_{x_0}^x G^\dagger(x_0)\|^{-1}$ is equal to the smallest nonzero singular value of $\tilde{\tau}_{x_0}^x G(x_0)$. Let $d(x_0, x)$ satisfy

$$d(x_0, x) < \frac{\|\tilde{\tau}_{x_0}^x G^\dagger(x_0)\|^{-1} - \|H_x^{-1}\|^{\frac{1}{2}} \|H_x\| \, \|[g_1^\Gamma \quad \cdots \quad g_m^\Gamma]\|}{(\|H_x^{-1}\| \|H_x\|)^{\frac{1}{2}} L_G}. \tag{20}$$

Substituting the right-hand side of (20) into (19) we get $\|\tilde{\tau}_{x_0}^x G(x_0) - G(x)\| \leq \|\tilde{\tau}_{x_0}^x G^\dagger(x_0)\|^{-1} = \sigma_r(\tilde{\tau}_{x_0}^x G^\dagger(x_0))$. Thus, by Weyl's inequality we have $\|\sigma_s(G(x)) - \sigma_s(\tilde{\tau}_{x_0}^x G(x_0))\| < \sigma_r(\tilde{\tau}_{x_0}^x G(x_0))$, i.e., $\sigma_s(G(x)) > 0$. Hence, $\mathrm{rank}(G(x)) \geq \mathrm{rank}(\tilde{\tau}_{x_0}^x G(x_0))$.

We now show that for any $d(x_0, x)$ which satisfies (20), $\mathrm{rank}(G(x)) = \mathrm{rank}(\tilde{\tau}_{x_0}^x G(x_0))$. If we substitute the left-hand side of (18) into the right-hand side of (20), we get

$$d(x_0, x) < \frac{\|\tilde{\tau}_{x_0}^x G^\dagger(x_0)\|^{-1} - \|\tilde{\tau}_{x_0}^x G(x_0) - \tau_{x_0}^x G(x)\|}{(\|H_x^{-1}\| \|H_x\|)^{\frac{1}{2}} L_G}.$$

This inequality implies

$$\|\tilde{\tau}_{x_0}^x G(x_0) - \tau_{x_0}^x G(x)\| <$$

$$\|\tilde{\tau}_{x_0}^x G^\dagger(x_0)\|^{-1} - (\|H_x^{-1}\| \|H_x\|)^{\frac{1}{2}} L_G d(x_0, x).$$

The term $(\|H_x^{-1}\| \|H_x\|)^{\frac{1}{2}} L_G d(x_0, x)$ is nonnegative, thus $\|\tilde{\tau}_{x_0}^x G(x_0) - \tau_{x_0}^x G(x)\| < \|\tilde{\tau}_{x_0}^x G^\dagger(x_0)\|^{-1}$ holds as well. Following similar steps as above, we see that $\mathrm{rank}(\tilde{\tau}_{x_0}^x G(x_0)) \geq \mathrm{rank}(\tau_{x_0}^x G(x_0))$. Recall we defined $G(x) = RH(x)$ by Assumption 1, so $\mathrm{Im}(G(x)) \subset \mathrm{Im}(R)$. Since $\mathrm{Im}(G(x_0)) = \mathrm{Im}(R)$, then $\mathrm{Im}(G(x)) \subset \mathrm{Im}(G(x_0))$. Thus, $\mathrm{rank}(G(x_0)) \geq \mathrm{rank}(G(x))$. On any complete Riemannian manifold, $\mathrm{rank}(G(x_0)) = \mathrm{rank}(\tau_{x_0}^x G(x_0))$. So we have $\mathrm{rank}(\tilde{\tau}_{x_0}^x G(x_0)) \geq \mathrm{rank}(\tau_{x_0}^x G(x_0)) = \mathrm{rank}(G(x_0)) \geq \mathrm{rank}(G(x))$. Hence, $\mathrm{rank}(\tilde{\tau}_{x_0}^x G(x_0)) = \mathrm{rank}(G(x))$. By Assumption 1, we now conclude that $\mathrm{Im}(\tilde{\tau}_{x_0}^x G(x_0)) = \mathrm{Im}(G(x)) = \mathrm{Im}(R)$ when (20) is satisfied. $\square$

We now compare this result to the case of control systems constrained in flat space.

Remark 1. *Let M be a flat manifold. Then, the domain which satisfies the condition that $\mathrm{Im}(\tilde{\tau}_{x_0}^x G(x_0)) = \mathrm{Im}(G(x))$ is identical to the domain in Lemma 1 of [1].*

The remark above is a consequence of $H_x = I$ and $\Gamma_{ij}^k = 0$. We emphasize that the remark above holds because the Euclidean space is itself a flat manifold, however, this remark proves that Lemma 1 from [1] is generalizable to any flat manifold. Now that we have determined the domain under which we will calculate guaranteed velocities, it is time to quantify an underapproximation of the *guaranteed velocity set* (GVS).

Theorem 1. *Let $f(x_0)$, $G(x_0)$, L_f, L_G, H_x, Γ_{ij}^k, and g_l^Γ for $l \in [m]$ be defined as above. Let $\gamma : [0,1] \to M$ define a geodesic curve from x_0 to x. Let $\tilde{\tau}$ define the parallel transport using the flat connection. Set $a(x) = (\|H_x^{-1}\| \|H_x\|)^{\frac{1}{2}} \|H_x\|^{\frac{1}{2}} \, \|[g_1^\Gamma \quad \cdots \quad g_m^\Gamma]\|$, $b(x) = (\|H_x^{-1}\| \|H_x\|)^{\frac{1}{2}} \left\| \sum_{i,j,k} \dot{\gamma}^i \Gamma_{ij}^k f^j(x_0) \vec{e}_k \right\|$, $c(x) = (\|H_x^{-1}\| \|H_x\|)^{\frac{1}{2}} \left(L_g + \|H_x\|^{-\frac{1}{2}} L_f \right) d(x_0, x)$ and*

$$\bar{d}(x_0, x) = \frac{\|\tilde{\tau}_{x_0}^x G^\dagger(x_0)\|^{-1} - a(x) - b(x)}{c(x)}.$$

If $d(x_0, x) \leq \bar{d}(x_0, x)$,

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$$\overline{\mathcal{V}}_x^{\mathcal{G}} = \mathbb{B}^n\left(\tilde{\tau}_{x_0}^x f(x_0); \alpha(x_0, x)\right) \cap \mathrm{Im}(\tilde{\tau}_{x_0}^x G(x_0)) \tag{21}$$

where $\overline{\mathcal{V}}_x^{\mathcal{G}} \in T_x M$, and

$$\begin{aligned}
\alpha(x_0, x) = &\|\tilde{\tau}_{x_0}^x G^\dagger(x_0)\|^{-1} - \\
&(\|H_x^{-1}\|\|H_x\|)^{\frac{1}{2}}\left(\|H_x\|^{\frac{1}{2}}\left\|[g_1^\Gamma \quad \cdots \quad g_m^\Gamma]\right\| + \right. \\
&\left.\left\|\sum_{i,j,k}\dot{\gamma}^i \Gamma_{ij}^k f^j(x_0)\vec{e}_k\right\| + \left(L_g + \|H_x\|^{-\frac{1}{2}}L_f\right)d(x_0, x)\right),
\end{aligned} \tag{22}$$

then $\overline{\mathcal{V}}_x^{\mathcal{G}} \subseteq \mathcal{V}_x^{\mathcal{G}}$.

Proof. Let $d \in \mathrm{Im}(\tilde{\tau}_{x_0}^x G(x_0)) = \mathrm{Im}(R)$ such that $\|d\| = 1$. We will prove that if $|k| \le \alpha(x_0, x)$, then

$$kd + \tilde{\tau}_{x_0}^x f(x_0) = f(x) + G(x)u \tag{23}$$

admits a solution $u \in \mathcal{U} = \mathbb{B}^m(0; 1)$ for any $(f, G) \in \mathcal{D}_{con}$. If we let $\eta(x_0, x)$ equal the right hand side of (20), we see that $\overline{d}(x_0, x) \le \eta(x_0, x)$. Hence, by Assumption 1 and Lemma 4, $\mathrm{Im}(G(x)) = \mathrm{Im}(R)$. We subtract both sides by $f(x) \in \mathrm{Im}(R)$. With $kd + \tilde{\tau}_{x_0}^x f(x_0) \in \mathrm{Im}(R)$, we have $kd + \tilde{\tau}_{x_0}^x f(x_0) - f(x) \in \mathrm{Im}(R)$. Hence, there exists a vector $\bar{u} \in \mathbb{R}^m$ such that $kd + \tilde{\tau}_{x_0}^x f(x_0) - f(x) = G(x)\bar{u}$. From here, we consider three cases, namely when $m < n$, $m = n$, and $m > n$.

We begin with the case when $m < n$. Through the rank nullity theorem [39], we can write $\bar{u} = u + u_2$ where $u \in \mathrm{Im}(G^T(x))$ and $u_2 \in \mathrm{Ker}(G(x))$. Thus, $G(x)\bar{u} = G(x)(u + u_2) = G(x)u$, and so $kd + \tilde{\tau}_{x_0}^x f(x_0) - f(x) = G(x)u$. We multiply both sides of $kd + \tilde{\tau}_{x_0}^x f(x_0) - f(x) = G(x)u$ on the left by $G^\dagger(x)$ and get $G^\dagger(x)(kd + \tilde{\tau}_{x_0}^x f(x_0) - f(x)) = G^\dagger(x)G(x)u$. The term $G^\dagger(x)G(x)u$ is the projection of u onto $\mathrm{Im}(G^T(x))$ [39]. Given that $u \in \mathrm{Im}(G^T(x))$, by definition of a projection, $G^\dagger(x)(kd + \tilde{\tau}_{x_0}^x f(x_0) - f(x)) = G^\dagger(x)G(x)u = u$. Taking the Euclidean norm of both sides, we show that if

$$\left\|G^\dagger(x)(kd + \tilde{\tau}_{x_0}^x f(x_0) - f(x))\right\| \le 1, \tag{24}$$

then $\|u\| \le 1$, i.e., $u \in \mathcal{U}$. For the instances where $m = n$ and $m > n$, $\mathrm{span}\{G(x)\} = T_x M$, so we can trivially follow a similar procedure to conclude that if (24) is satisfied, there exists at least one $u \in \mathcal{U}$ which serves as a solution to (23).

The product and triangle inequality for norms gives us $\left\|G^\dagger(x)(kd + \tilde{\tau}_{x_0}^x f(x_0) - f(x))\right\| \le \|G^\dagger(x)\|(|k|\|d\| + \|\tilde{\tau}_{x_0}^x f(x_0) - f(x)\|)$. Hence, any k which satisfies

$$\left\|G^\dagger(x)\right\|(|k|\|d\| + \left\|\tilde{\tau}_{x_0}^x f(x_0) - f(x)\right\|) \le 1 \tag{25}$$

satisfies (24) and admits a solution $u \in \mathcal{U}$. We may substitute $\|d\| = 1$; multiplying on the left by $\|G^\dagger(x)\|^{-1}$ and subtracting $\|\tilde{\tau}_{x_0}^x f(x_0) - f(x)\|$ from both sides, we see that any k which satisfies

$$|k| \le \left\|G^\dagger(x)\right\|^{-1} - \left\|\tilde{\tau}_{x_0}^x f(x_0) - f(x)\right\| \tag{26}$$

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also satisfies (25) and thus admits a solution $u \in \mathcal{U}$.

From (19) in the proof of Lemma 4 we have an upper bound on $\|G(x) - \tilde{\tau}_{x_0}^x G(x_0)\|$. We then apply the reverse triangle inequality and add $\|\tilde{\tau}_{x_0}^x G(x_0)\|$ to both sides of (19) to get

$$\|G(x)\| \leq \|\tilde{\tau}_{x_0}^x G(x_0)\| +$$
$$\left(\frac{\|H_x\|}{\|H_x^{-1}\|^{-1}}\right)^{\frac{1}{2}} \left(\|H_x\|^{\frac{1}{2}} \left\| \begin{bmatrix} g_1^\Gamma & \cdots & g_m^\Gamma \end{bmatrix} \right\| + L_G d(x_0, x)\right). \tag{27}$$

Note that $\|G^\dagger(x)\|^{-1}$ is equal to the smallest singular value of $G(x)$ and that $\|G^\dagger(x)\|^{-1} = \|\tilde{\tau}_{x_0}^x G^\dagger(x_0)\|^{-1}$ by metric compatibility. Thus, we can apply Weyl's inequality for singular values [42] to get

$$\|G^\dagger(x)\|^{-1} \geq \|\tilde{\tau}_{x_0}^x G^\dagger(x_0)\|^{-1} -$$
$$\left(\frac{\|H_x\|}{\|H_x^{-1}\|^{-1}}\right)^{\frac{1}{2}} \left(\|H_x\|^{\frac{1}{2}} \left\| \begin{bmatrix} g_1^\Gamma & \cdots & g_m^\Gamma \end{bmatrix} \right\| + L_G d(x_0, x)\right). \tag{28}$$

We also want to bound $|\tilde{\tau}_{x_0}^x f(x_0) - f(x)|_{h_x}$ in terms we know from our assumptions. It holds that $|\tilde{\tau}_{x_0}^x f(x_0) - f(x)|_{h_x} \leq |\tilde{\tau}_{x_0}^x f(x_0) - \tau_{x_0}^x f(x_0)|_{h_x} + |\tau_{x_0}^x f(x_0) - f(x)|_{h_x}$. From Assumption 2, we know that $|\tau_{x_0}^x f(x_0) - f(x)|_{h_x} \leq L_f d(x_0, x)$. The left-hand side of inequality (4) the inequality above imply that $\|\tau_{x_0}^x f(x_0) - f(x)\| \leq \|H_x^{-1}\|^{\frac{1}{2}} L_f d(x_0, x)$. Repeating a similar process as in the proof of Lemma 4, we have that

$$|\tilde{\tau}_{x_0}^x f(x_0) - \tau_{x_0}^x f(x_0)|_{h_x} \leq \left| \sum_{i,j,k} \dot{\gamma}^i \Gamma_{ij}^k f^j(x_0) \vec{e}_k \right|_{h_x}$$

where vectors $\vec{e}_k$ form the basis for $T_x M$. Again, the inequalities in (4) with the inequality above imply that

$$\|\tilde{\tau}_{x_0}^x f(x_0) - \tau_{x_0}^x f(x_0)\| \leq \|H_x^{-1}\|^{\frac{1}{2}} \|H_x\|^{\frac{1}{2}} \left\| \sum_{i,j,k} \dot{\gamma}^i \Gamma_{ij}^k f^j(x_0) \vec{e}_k \right\|$$

holds. With both terms bounded, we can conclude that

$$\|\tilde{\tau}_{x_0}^x f(x_0) - f(x)\| \leq$$
$$\|H_x^{-1}\|^{\frac{1}{2}} \left(\|H_x\|^{\frac{1}{2}} \left\| \sum_{i,j,k} \dot{\gamma}^i \Gamma_{ij}^k f^j(x_0) \vec{e}_k \right\| + L_f d(x_0, x)\right). \tag{29}$$

From (28) and (29), if k satisfies

$$|k| \leq \alpha(x_0, x), \tag{30}$$

then it satisfies (26) and by extension (24), thereby admitting a solution $u \in \mathcal{U}$ and proving the claim. $\qquad \square$
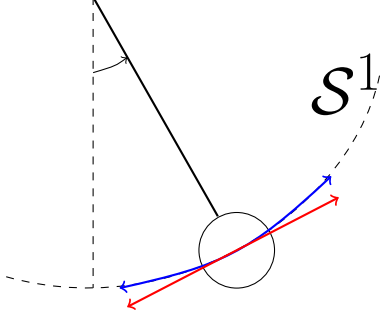
Figure 2: The set of achievable velocities $\dot{x} \in T_x M$ in red and the trajectories formed using the exponential map for $\mathcal{S}^1$ in blue.

From the proof, we see that the domain under which we can calculate $\overline{\mathcal{V}}_x^{\mathcal{G}}$ is not further restricted by the domain derived in Lemma 4 under which Assumption 1 and 2 remain consistent. In [1], we derived a geometrically similar ball underapproximation for systems constrained on Euclidean manifolds. For comparison with results in [1], we provide this quick remark.

Remark 2. *Let M be a flat manifold. Then $\overline{\mathcal{V}}_x^{\mathcal{G}}$ is equal to the set calculated in Theorem 1 of [1].*

Much like Remark 1, Remark 2 follows because $H_x = I$ and $\Gamma_{ij}^k = 0$ for flat manifolds. Remark 2 is stating that the underapproximation from Theorem 1 of [1] should hold for any nonlinear control-affine system operating on any flat manifold, not just the Euclidean space. Now that we have guaranteed underapproximations of the GVS, we describe in the next section how to use such sets to determine an underapproximation of the GRS which lies on a complete Riemannian manifold.

IV   Reachable Set

To calculate the guaranteed reachable set (GRS) of an unknown system, we first use Theorem 1 from Section III to determine the guaranteed velocity sets for different $x \in M$. Then, similar to [1, 2], we leverage these sets to find a control system whose reachable set is an underapproximation of the GRS.

We first define an ordinary differential inclusion based on the underapproximation of the GVS in Theorem 1. Then, we formulate a control system whose set of velocities are equal the right hand side of the ODI. The reachable set of this formulated control systems contained in the GRS by Proposition 1. Theorem 1 and Proposition 1 show that the reachable set of

$$\dot{x} \in \overline{\mathcal{V}}_x^{\mathcal{G}}, \quad x(0) = x_0 \tag{31}$$

is a subset of $\mathcal{R}_x^{\mathcal{G}}(T, x_0)$.

We follow similar steps as in [1, 2]: Let $\overline{d}(x_0, x)$ and $\alpha(x_0, x)$ be as defined in Theorem 1. Analogous to the interpretation of the dynamics (1) as an ODI (11), we can interpret (31) as a control system

$$\dot{x} = a + g(x_0, x)u, \quad x(0) = x_0, \tag{32}$$

on $\{x \mid d(x_0, x) \leq \bar{d}(x_0, x)\}$, with $a = \tilde{\tau}^x_{x_0} f(x_0)$, $u \in \mathbb{B}^n(0; 1)$, and $g(s_0, s) = \alpha(x_0, x)$ if $d(s_0, s) \leq \bar{d}(x_0, x)$. We thus obtain the following result.

Theorem 2. *Let* $\overline{\mathcal{R}}(T, x_0)$ *be defined as the reachable set of* (32) *at time* T. *Then,* $\overline{\mathcal{R}}(T, x_0) \subset \mathcal{R}^{\mathcal{G}}(T, x_0)$.

Proof. Proposition 1 and Theorem 1 prove the claim. $\qquad\square$

Now that we have a control system whose solutions are contained in the guaranteed reachable set, we want to determine how determine such solutions numerically on a manifold. Existing numerical methods such as multistep algorithms and sympletic integration techniques like Runge-Kutta [43, 44] for manifolds can be used to determine such a reachable set. We emphasize that systems are often described by ordinary differential equations in Euclidean spaces which constrain their dynamics to a naturally embedded manifold — see section I.1, I.2, and I.3 in [45] — or the use of differential algebraic constraint equations [46]. The solutions to such control systems are computed numerically on Euclidean spaces and projected onto the manifolds.

We employ the techniques mentioned above to numerically determine the solution to our derived control system whose reachable set is contained in the GRS. For the aforementioned projection methods, numerically solving for the solution to a system evolving on a manifold requires the extra step of mapping velocities in the tangent space to their corresponding trajectories on M. Such a mapping is provided via the **exponential map** [9, 47]. We illustrate the exponential map on the example of a pendulum constrained on $\mathcal{S}^1$ in Fig. 2 and formally define it below.

Definition 10 ((Geodesic) exponential map)**.** *Let* M *be a Riemannian manifold and* $x \in M$. *The exponential map*

$$\exp_{x_0} : \ U \subseteq T_{x_0}M \to M : \ \dot{x} \to x(1)$$

at x_0 *assigns to* $\dot{x} \in T_{x_0}M$ *the solution at time* $t = 1$ *to the geodesic equation with initial state* x_0 *and initial velocity* $\dot{x}$ *where* $x(0) = x_0$ *and* $\nabla_{\dot{x}}\dot{x} = 0$.

Recall that M is a connected Riemannian manifold. Thus, by the Hopf-Rinow Theorem [9], the exponential map is defined on all T_xM for all $x \in M$. We leverage the exponential map to calculate a set of geodesic curves on M where each curve serves as a viable trajectory contained in the guaranteed reachable set.

In the next section, we demonstrate by example how to apply the results in Sections III and IV to calculate the guaranteed reachable set of an unknown pendulum system; this example is simple enough to solve using Euler's method augmented with exponential mapping and is primarily included to demonstrate how to implement the theory intuitively. We then apply our theory to a more complex example of a system on an abstract manifold $SO(3)$. Trajectories of the known system and the guaranteed underapproximation of its reachable set are compared to illustrate the theory's efficacy.

V Numerical Examples

In this section, we demonstrate how to calculate a guaranteed reachable set provably contained within the true reachable set of two unknown systems. We first apply results an example of a heavily damped pendulum which lies on a one-dimensional sphere $\mathcal{S}^1$. We present the first example to illustrate the general pipeline and show how the novel theory can be applied on a system which operates on a non-Euclidean manifold. For the second example, we determine an underapproximation of the GRS for a rotational system which lies in $SO(3)$. The application in question is to determine whether certain rotations can be achieved within a finite time horizon given initial conditions. We emphasize that the structure of $SO(3)$ is significantly different from that of the Euclidean space. Consequently, it is not possible to solve this problem using the results in [1].
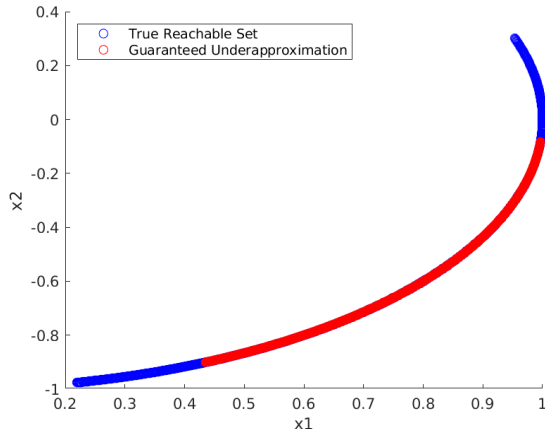
V-A Pendulum

Figure 3: The blue and red show the true reachable set and underapproximated guaranteed reachable set $\overline{\mathcal{R}}(T, \theta_0)$ of (33) respectively for $T = 1$ and $\theta_0 = \pi/4$.

In this first example, we are concerned with determining the set of provably attainable positions of the end effector of a fully actuated pendulum constrained on $\mathcal{S}^1$. With knowledge from Assumption 1 and 2 we can calculate an underapproximation of the GRS. We calculate this underapproximation numerically by implementing the commonly used Euler Method [48] and mapping x_{t+1} at every time step onto the manifold using the exponential map corresponding to $\mathcal{S}^1$.

A pendulum with known dynamics is often modeled as a series of two ordinary differential equations, thus making it a second order system operating on $\mathcal{S}^1 \times \mathbb{R}$ with the following ODE [49]:

$$ml^2\ddot{\theta} + b\dot{\theta} = u - mgl\sin(\theta).$$

For a highly damped pendulum when $b\dot{\theta} \gg ml^2\ddot{\theta}$, which occurs when $b\sqrt{\frac{l}{g}} \gg ml^2$, we have an approximation through a first order system [50]:

$$b\dot{\theta} = u - mgl\sin\theta.$$

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Substituting $m = 0.51\ kg$, $g = 9.8\ m/s^2$, $l = 0.1\ m$, and $b = 1\ s^{-1}$ gives us a highly damped system

$$\dot{\theta} = -\frac{1}{2}\sin\theta + u \tag{33}$$

where $u \in [-1, 1]$. We consider the system with the initial condition $\theta_0 = \pi/4$.

To determine upper bounds L_f and L_G on this system, we assume we can leverage design information and physical laws to determine that $L_f = 1.5$ and $L_G = 0$. The manifold $\mathcal{S}^1$ is a flat manifold [9] and thus $\Gamma_{11}^1 = 0$ and the metric tensor $H_\theta = H_\theta^{-1} = 1$ for all $\theta \in \mathcal{S}^1$. With the initial condition $\theta_0 = \pi/4$, we have $f(\theta_0) = -\sqrt{2}/4$. Additionally, $\|G^\dagger(\theta_0)\|^{-1} = 1$, and by metric compatibility it follows that $\|\tilde{\tau}_{\theta_0}^\theta G^\dagger(\theta_0)\|^{-1} = 1$. We plug these into the expression (22) to calculate $\mathcal{V}_\theta^\mathcal{G}$ from (21):

$$\begin{aligned}\mathcal{V}_\theta^\mathcal{G} &= \mathbb{B}^1(\tilde{\tau}_{\theta_0}^\theta f(\theta_0); \|\tilde{\tau}_{\theta_0}^\theta G^{-1}(\theta_0)\|^{-1} - (L_G + L_f)\,d(\theta_0, \theta)).\\ &= \mathbb{B}^1\left(-\frac{\sqrt{2}}{4}; 1 - 1.5d(\theta_0, \theta)\right).\end{aligned} \tag{34}$$

By Theorem 2, we construct the control system

$$\dot{\theta} = \frac{4 - \sqrt{2}}{4} - 1.5d(\theta_0, \theta)u \tag{35}$$

whose set of velocities is the same as those in (34).

We utilize (35) to numerically solve for the GRS. The exponential map for $\mathcal{S}^1$ is given as

$$\exp_\theta(\dot{\theta}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta + \dot{\theta}\delta t) \\ \sin(\theta + \dot{\theta}\delta t) \end{bmatrix} \tag{36}$$

where $\delta t = 0.001s$ is the chosen time step. At each time step, we ran a Monte Carlo simulation [51] where the velocity from (35) is mapped to $\mathcal{S}^1$ using the exponential map for $\mathcal{S}^1$ (36) as stated in the previous section. The process is then repeated to calculate $\overline{\mathcal{R}}(T, \theta_0)$. By observing Fig. 3, we see that, consistent with the theoretical results, $\overline{\mathcal{R}}(T, \theta_0) \subset \mathcal{R}^\mathcal{G}(T, \theta_0)$.

V-B The Set of all 3 x 3 Real Orthogonal Matrices

We present an example of a control system which operates on $SO(3) = \{N \in \mathbb{R}^{3\times3} \mid N^T N = I, \det(N) = 1\}$, the special orthogonal group whose elements form the set of all real orthogonal matrices with a determinant of 1. Let $L(SO(3))$ be the algebra consisting of all 3×3 antisymmetric matrices. A basis for $L(SO(3))$ [3, Example 8.1] is given by the matrices

$$K_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

and

$$K_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$
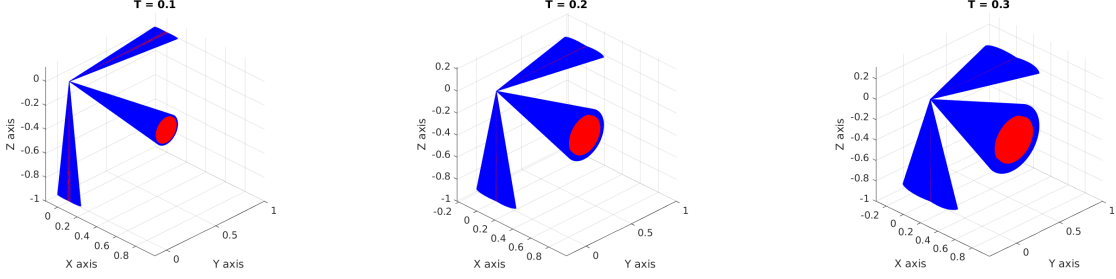
Figure 4: True reachable set (blue) with the underapproximation $\overline{\mathcal{R}}(T, x_0)$ (red) numerically calculated for $T \in \{0.1, 0.2, 0.3\}$ seconds. Reachable sets display the set of all reachable orthonormal triads with initial condition X_0 up to time T using the fully known dynamics (37) (blue) and the underapproximated system (32) (red).

For any point $X \in SO(3)$, we take the basis of $T_X M$ to be $\{K_x X, K_y X, K_z X\}$. We consider

$$X(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

which corresponds to a rotation of $\pi/2$ around the unit vector $(0, 1, 0)$. To check $X(0) \in SO(3)$, we verify that $\det(X(0)) = 1$ and $X^T(0)X(0) = I$.

We consider an unknown system evolving on $SO(3)$ with dynamics

$$\dot{x}(t) = f(x(t)) + G(x(t))u, \quad x(0) = x_0,$$
$$\dot{X}(t) = \dot{x}_1 K_x X(t) + \dot{x}_2 K_y X(t) + \dot{x}_3 K_z X(t), \tag{37}$$

where $x \in \mathbb{R}^3$, $u \in \mathbb{B}^2(0; 1)$, $f = 0$, and $G(x(t)) = \begin{bmatrix} 0 & 0 & 1 + 0.5x_3 \\ 0 & 1 & 0 \end{bmatrix}^T$. We operate on the open set $U \in SO(3)$ and $\varphi : U \to \mathbb{R}^3$ such that (U, φ) is a chart in $SO(3)$ where $\varphi(U) = \{(\psi, \theta, \phi) : 0 < \psi < 2\pi, 0 < \theta < \pi, 0 < \phi < 2\pi\}$. The initial condition $X(0)$ corresponds to the initial condition $x_0 = (0, \pi/2, 0)$. Let $\{e_i : i = 1, 2, 3\}$ be an orthonormal triad in $SO(3)$. Each rotation X on $SO(3)$ acts on the triad $\{e_i\}$ to produce another orthonormal triad $\{Xe_i : i = 1, 2, 3\}$. Indeed, X is represented under the basis $\{e_i\}$ by the matrix $\begin{bmatrix} Xe_1 & Xe_2 & Xe_3 \end{bmatrix}$ with Xe_i as the ith column [52, Section 1.7]. Local coordinates $x \in \mathbb{R}^3$ represent the Euler angles (ψ, θ, ϕ) which describe the orientation of the orthonormal triad in $SO(3)$.

There are three axes around which $X(t) \in SO(3)$ can rotate, hence this is a 3-degree of freedom system. We have $\mathcal{U} = \mathbb{B}^2(0; 1)$ with $f = 0$, so we can conclude that this is an underactuated control system consistent with Assumption 1. We can thus calculate the guaranteed set of reachable states using the knowledge from Assumption 2 without knowledge of the true dynamics using the novel theory developed in previous sections. To determine an underapproximation of the GVS, we want to calculate

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$$\overline{\mathcal{V}}_x^{\mathcal{G}} = \mathbb{B}^3(0; \alpha(x_0, x)) \cap \text{Im}(\tilde{\tau}_{x_0}^x G(x_0)), \tag{38}$$

so we need to solve for $\alpha(x_0, x)$. It follows that we need to quantify $\tilde{\tau}_{x_0}^x f(x_0)$, $\|\tilde{\tau}_{x_0}^x G^\dagger(x_0)\|^{-1}$, $d(x_0, x)$, L_f, L_G, $\|H_x^{-1}\|^{-1}$, $\|H_x\|$, and Γ_{ij}^k to calculate the term $\alpha(x_0, x)$ defined in (22) for any $X \in SO(3)$. By Assumption 2, we know $f(x_0) = 0$ and so trivially $\tilde{\tau}_{x_0}^x f(x_0) = 0$ and $L_f = 0$. Since initial conditions $(x_1(0), x_2(0), x_3(0)) = (\psi_0, \theta_0, \phi_0) = (0, \pi/2, 0)$, it follows that $\|\tilde{\tau}_{x_0}^x G^\dagger(x_0)\|^{-1} = 1$. Note that by metric compatibility, $\|\tilde{\tau}_{x_0}^x G^\dagger(x_0)\|^{-1}$ is invariant under the transport $\tilde{\tau}$.

On the open set U defined earlier, $d(x, x_0)$ using the Euler angles (ψ, θ, ϕ) is defined as the misorientation angle — the angle between two elements $X_0, X \in SO(3)$. From [52, Section 10.2] we can define the misorientation of X with respect to X_0 given below

$$d(x_0, x) = \cos^{-1}\left(\frac{1}{2}\text{Tr}(X_0 X^T) - 0.5\right).$$

In [52, Section 10.3], for the neighborhood U we calculate the metric tensor as

$$H_x = \begin{bmatrix} 1 & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & 1 \end{bmatrix}.$$

From here, we calculate the inverse metric tensor to be

$$H_x^{-1} = \begin{bmatrix} \frac{1}{\sin^2\theta} & 0 & -\frac{\cos\theta}{\sin^2\theta} \\ 0 & 1 & 0 \\ -\frac{\cos\theta}{\sin^2\theta} & 0 & \frac{1}{\sin^2\theta} \end{bmatrix}.$$

With H_x and H_x^{-1} we can calculate $\Gamma_{ij}^k(x)$ using equation (8). Lastly, for some neighborhood $V \subset U$ on which we operate, by Assumption 2 we are given $L_{g_1} = 0.65$, $L_{g_2} = 0$, $\|H_x\| < 1.2$, $\|H_x^{-1}\|^{-1} > 0.8$ for all $x \in V$. From Lemma 3, we calculate a valid upper bound for equation (9) as $L_G = n\|H_x^{-1}\|\|H_x\|^{\frac{1}{2}}L_{g_1} \approx 2.7$.

The corresponding GRS calculated using $\mathcal{V}_x^{\mathcal{G}}$ from (38) is shown in Fig. 4 for $T \in \{0.1, 0.2, 0.3\}$. Fig. 4 shows all possible orientations of an orthonormal triad with initial condition X_0 within a finite time horizon for the known control system (37) and the underapproximated control system (32). The blue represents all possible orientations using the known control system (37) while the red represent the same information gathered using solely the novel theory presented in this paper without knowledge of (37). Consistent with the theory presented in this paper, the GRS calculated using the presented theory is contained within the interior of the union of all blue triads which represent the reachable set calculated using the known dynamics of system (37).

VI   Conclusion and Future Work

This paper presents a novel approach for determining an underapproximation of an unknown nonlinear control system's reachable set which lies on any complete Riemannian manifold. By

assuming a nonlinear control-affine structure and knowledge of the dynamics at a single point, we can produce a set $\overline{\mathcal{R}}(T, x_0) \subseteq \mathcal{R}^{\mathcal{G}}(T, x_0)$ which consists of provably reachable trajectories for the unknown system. The underapproximation relies on an intermediate approximation of the guaranteed set of reachable states by an ODI $\dot{x} \in \mathcal{V}_x^{\mathcal{G}}$ where its right-hand side is a set of guaranteed velocities for the unknown nonlinear system. We applied novel theoretical results on two systems — a pendulum and a rotational control system existing in the special orthogonal group $SO(3)$. The first example presented a high-level illustration of how the theory could be practically implemented on systems whose trajectories lie on any complete Riemannian manifold. We also demonstrate the efficacy of our approach on a system that lies on the more abstract manifold of real orthogonal matrices $SO(3)$ whose structure is not diffeomorphic to the Euclidean space.

A natural area of future work is to increase the knowledge of the system dynamics beyond Riemannian Lipschitz bounds, i.e., to utilize additional information to increase the size of our underapproximations without knowledge of the true system dynamics. Instead of maximizing the knowledge from just one trajectory, we can incorporate the knowledge from many trajectories. This can potentially generalize results to include all underactuated nonlinear control-affine systems by altering the space in which the unknown dynamics lie. Another natural extension of this research is to extend the results beyond the deterministic domain and consider how similar methods can apply to models constructed by neural nets with parametric noise. This noise would pertain to modeling uncertainty, possibly due to to error in system identification for unknown systems using multiply trajectories. Such efforts would provide certifiable capabilities for neural network system models which often lack robust provable performance guarantees.

VII   Acknowledgements

References

[1] T. Shafa and M. Ornik, "Reachability of nonlinear systems with unknown dynamics," *IEEE Transactions on Automatic Control*, vol. 68, no. 4, pp. 2407–2414, 2022.

[2] ——, "Maximal ellipsoid method for guaranteed reachability of unknown fully actuated systems," in *2022 IEEE 61st Conference on Decision and Control (CDC)*. IEEE, 2022, pp. 5002–5007.

[3] V. Jurdjevic and H. J. Sussmann, "Control systems on Lie groups," *Journal of Differential Equations*, vol. 12, no. 2, pp. 313–329, 1972.

[4] C. Altafini, "Feedback stabilization of isospectral control systems on complex flag manifolds: application to quantum ensembles," *IEEE Transactions on Automatic Control*, vol. 52, no. 11, pp. 2019–2028, 2007.

[5] J. Zhu, E. Trélat, and M. Cerf, "Geometric optimal control and applications to aerospace," *Pacific Journal of Mathematics for Industry*, vol. 9, no. 1, pp. 1–41, 2017.

[6] G. S. Chirikjian, *Stochastic models, information theory, and Lie groups, volume 2: Analytic methods and modern applications.* Springer Science & Business Media, 2011, vol. 2.

[7] H. Schättler and U. Ledzewicz, *Geometric optimal control: theory, methods and examples.* Springer, 2012, vol. 38.

[8] S. Liu, Y. Fan, and M.-A. Belabbas, "Affine geometric heat flow and motion planning for dynamic systems," *IFAC-PapersOnLine*, vol. 52, no. 16, pp. 168–173, 2019.

[9] J. Jost, *Riemannian geometry and geometric analysis.* Springer, 2008.

[10] R. L. Bishop and S. I. Goldberg, *Tensor analysis on manifolds.* Courier Corporation, 2012.

[11] M. Spivak, *Calculus on manifolds: a modern approach to classical theorems of advanced calculus.* CRC Press, 2018.

[12] H. Benninghoff, T. Boge, and F. Rems, "Autonomous navigation for on-orbit servicing," *KI-Künstliche Intelligenz*, vol. 28, pp. 77–83, 2014.

[13] Y. Fan, S. Liu, and M.-A. Belabbas, "Geometric heat flow method for legged locomotion planning," *IEEE Control Systems Letters*, vol. 5, no. 3, pp. 941–946, 2020.

[14] M. Spong, K. Khorasani, and P. Kokotovic, "An integral manifold approach to the feedback control of flexible joint robots," *IEEE Journal on Robotics and Automation*, vol. 3, no. 4, pp. 291–300, 1987.

[15] A. A. Kilin, E. N. Pivovarova, and T. B. Ivanova, "Spherical robot of combined type: Dynamics and control," *Regular and chaotic dynamics*, vol. 20, pp. 716–728, 2015.

[16] R. W. Brockett, "Nonlinear systems and differential geometry," *Proceedings of the IEEE*, vol. 64, no. 1, pp. 61–72, 1976.

[17] A. Isidori, *Nonlinear Control Systems.* Springer Science & Business Media, 2013.

[18] I. M. Mitchell and C. J. Tomlin, "Overapproximating reachable sets by Hamilton-Jacobi projections," *Journal of Scientific Computing*, vol. 19, no. 1, pp. 323–346, 2003.

[19] H. Kong, E. Bartocci, and T. A. Henzinger, "Reachable set over-approximation for nonlinear systems using piecewise barrier tubes," in *International Conference on Computer Aided Verification*, 2018, pp. 449–467.

[20] M. Ornik, S. Carr, A. Israel, and U. Topcu, "Control-oriented learning on the fly," *IEEE Transactions on Automatic Control*, vol. 65, no. 11, pp. 4800–4807, 2019.

[21] J. Ding, E. Li, H. Huang, and C. J. Tomlin, "Reachability-based synthesis of feedback policies for motion planning under bounded disturbances," in *2011 IEEE International Conference on Robotics and Automation*, 2011, pp. 2160–2165.

[22] S. V. Rakovic, E. C. Kerrigan, D. Q. Mayne, and J. Lygeros, "Reachability analysis of discrete-time systems with disturbances," *IEEE Transactions on Automatic Control*, vol. 51, no. 4, pp. 546–561, 2006.

[23] S. Mohan and R. Vasudevan, "Convex computation of the reachable set for hybrid systems with parametric uncertainty," in *2016 American control conference*, 2016, pp. 5141–5147.

[24] M. Althoff, O. Stursberg, and M. Buss, "Reachability analysis of nonlinear systems with uncertain parameters using conservative linearization," in *2008 47th IEEE Conference on Decision and Control*, 2008, pp. 4042–4048.

[25] B. Wingo, M. Murtaza, V. Azimi, and S. Hutchinson, "Adaptively robust control policy synthesis through Riemannian motion policies," *IEEE Control Systems Letters*, vol. 6, pp. 31–36, 2020.

[26] S. Calinon, "Gaussians on Riemannian manifolds: Applications for robot learning and adaptive control," *IEEE Robotics & Automation Magazine*, vol. 27, no. 2, pp. 33–45, 2020.

[27] H. Sussmann, "Reachability by means of nice controls," in *26th IEEE Conference on Decision and Control*, vol. 26.   IEEE, 1987, pp. 1368–1373.

[28] V. Ayala, H. Román-Flores, and A. Da Silva, "About the continuity of reachable sets of restricted affine control systems," *Chaos, Solitons & Fractals*, vol. 94, pp. 37–43, 2017.

[29] O. Thapliyal and I. Hwang, "Approximating reachable sets for neural network-based models in real time via optimal control," *IEEE Transactions on Control Systems Technology*, 2023.

[30] A. Devonport and M. Arcak, "Data-driven reachable set computation using adaptive gaussian process classification and monte carlo methods," in *2020 American Control Conference (ACC)*, 2020, pp. 2629–2634.

[31] J.-P. Aubin and A. Cellina, *Differential Inclusions: Set-Valued Maps and Viability Theory*.   Springer Science & Business Media, 2012.

[32] A. Bressan and B. Piccoli, *Introduction to the mathematical theory of control*.   American institute of mathematical sciences Springfield, 2007, vol. 1.

[33] G. V. Smirnov, *Introduction to the Theory of Differential Inclusions*.   American Mathematical Society, 2002.

[34] R. D. Canary, A. Marden, and D. B. Epstein, *Fundamentals of Hyperbolic Manifolds: Selected expositions*.   Cambridge University Press, 2006.

[35] H. Nijmeijer and A. Van der Schaft, *Nonlinear dynamical control systems*. Springer, 1990, vol. 464, no. 2.

[36] R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, tensor analysis, and applications*. Springer Science & Business Media, 2012, vol. 75.

[37] T. Frankel, *The geometry of physics: an introduction*. Cambridge university press, 2011.

[38] M. P. Do Carmo and J. Flaherty Francis, *Riemannian Geometry*. Springer, 1992.

[39] G. Strang, *Introduction to Linear Algebra*. Wellesley-Cambridge Press, 2016.

[40] B. O'neill, *Elementary Differential Geometry*. Elsevier, 2006.

[41] M. Ornik, "Guaranteed reachability for systems with unknown dynamics," in *2020 59th IEEE Conference on Decision and Control (CDC)*. IEEE, 2020, pp. 2756–2761.

[42] G. W. Stewart, "Perturbation theory for the singular value decomposition," Tech. Rep., 1998.

[43] P. E. Crouch and R. Grossman, "Numerical integration of ordinary differential equations on manifolds," *Journal of Nonlinear Science*, vol. 3, pp. 1–33, 1993.

[44] C. J. Budd and A. Iserles, "Geometric integration: numerical solution of differential equations on manifolds," *Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, vol. 357, no. 1754, pp. 945–956, 1999.

[45] E. Hairer, "Solving differential equations on manifolds," 2011.

[46] E. A. Cross and I. M. Mitchell, "Level set methods for computing reachable sets of systems with differential algebraic equation dynamics," in *2008 American Control Conference*, 2008, pp. 2260–2265.

[47] P.-A. Absil, R. Mahony, and R. Sepulchre, *Optimization Algorithms on Matrix Manifolds*. Princeton University Press, 2008.

[48] M. Kamruzzaman and M. C. Nath, "A comparative study on numerical solution of initial value problem by using Euler's method, modified Euler's method and Runge–Kutta method," *Journal of computer and mathematical sciences*, vol. 9, no. 5, pp. 493–500, 2018.

[49] G. L. Baker and J. A. Blackburn, *The pendulum: a case study in physics*. OUP Oxford, 2008.

[50] R. Tedrake, "Underactuated robotics: Learning, planning, and control for efficient and agile machines course notes for mit 6.832," *Working draft edition*, vol. 3, no. 4, p. 2, 2009.

[51] C. Z. Mooney, *Monte Carlo Simulation.*   Sage, 1997.

[52] C.-S. Man, "Crystallographic texture and group representations," *Journal of Elasticity*, vol. 149, no. 1, pp. 3–445, 2022.